Reactive Planar Spanner Construction in Wireless Ad Hoc and Sensor Networks

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***Abstract*—Within reactive topology control, a node determines its adjacent edges of a network subgraph without prior knowledge of its neighborhood. The goal is to construct a local view on a topology which provides certain desired properties such as planarity. During algorithm execution, a node, in general, is not allowed to determine all its neighbors of the network graph.**

There are well-known reactive algorithms for computing planar subgraphs. However, the subgraphs obtained do not have constant Euclidean spanning ratio. This means that routing along these subgraphs may result in potentially long detours.

So far, it has been unknown if planar spanners can be constructed reactively. In this work, we show that at least under the unit disk network model, this is indeed possible, by proposing an algorithm for reactive construction of the partial Delaunay triangulation, which recently turned out to be a spanner. Furthermore, we show that our algorithm is message-optimal as a node will only exchange messages with nodes that are also neighbors in the spanner. The algorithm's presentation is complemented by a rigorous proof of correctness.

***Index Terms*—Reactive topology control, Euclidean spanner, partial Delaunay triangulation, localized algorithm.**

I. Introduction

Localized topology control is defined as a network node's task of determining its incident edges of the network subgraph, guaranteeing certain graph properties. The most prominent properties are planarity and the Euclidean spanning ratio. The former applies if any two edges cross at most in their endpoints, whereas the latter refers to the maximum ratio of the shortest Euclidean path between any two nodes in the subgraph, to the shortest Euclidean path between these nodes in the original network graph. Especially with respect to wireless sensor networks, where energy is a valuable and limited resource, a localized topology control algorithm should conserve message transmissions.

However, for a node to determine certain network neighbors, some messages have to be exchanged. A simple approach is beaconing, i.e., gathering messages from all network neighbors, prior to the application of some edge selection rule for subgraph construction. This is contrasted by the algorithmically more sophisticated beaconless approach, in which the best case only relevant neighbors answer (i.e., those that belong to the desired subgraph) and thus, as much messages as possible are saved.

Depending on the application, topology control can be executed proactively (periodically) or on demand. The latter is often referred to as *reactive*, even if beacons are used or full neighborhood knowledge is assumed. To our understanding, reactive actually refers to an on-demand protocol execution, in which the protocol neither uses beacons, nor assumes any given knowledge on the neighborhood. Henceforth, we use the term in this sense.

Localized topology control applies, for example, in greedy recovery routines of geographic routing algorithms. A prominent recovery routine is Face routing [1], which has been proven to guarantee the message's delivery if the message is forwarded along the edges of a planar subgraph of the network graph [2]. Since the subgraph's spanning ratio has a direct impact on the length of potential routing paths selected by the routing algorithm, the spanning ratio provided should be as small as possible and most importantly, independent of the number of network nodes, i.e., it should be constant.

At least in the unit disk model, where two network nodes are connected by an edge iff the Euclidean distance between them does not exceed some predefined unit distance, beacon-based planar spanner construction is rather simple: For a node it suffices to compute its incident neighbors in the *partial Delaunay triangulation* [3], which recently has been proven to have a constant Euclidean spanning ratio [4]. In contrast to beacon-based topology control, the currently best reactive approach [5] is to construct the Gabriel graph [6], which has spanning ratio of $\Theta(\sqrt{n})$ [7].

In this work, we essentially show how to close this gap concerning spanning ratios between beacon-based and reactive localized topology control, by proposing the first reactive protocol for planar spanner construction under the unit disk model. This positively answers the open question of whether spanners can be computed reactively (see e.g., [8], [9]). We now review important related work in more detail and subsequently discuss our contributions in that context.

A. One-hop localized subgraph constructions

Considering one-hop localized constructions, the *partial Delaunay triangulation* (PDT) [3] is currently the densest

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known planar subgraph of the unit disk graph (UDG). It is a supergraph of the UDG intersected with the Gabriel graph (GG) [6], where the latter contains an edge (u, v) iff the circle that has (u, v) as its diameter (called *Gabriel circle*) is empty of other nodes. PDT contains an edge between two nodes u and v iff (u, v) is a Gabriel edge or for the node w maximizing the angle $\angle uwv$ among all nodes within the Gabriel circle around (u, v), the unique circle through u, v, w is empty of other nodes and entirely contained by the unit disk centered at u (see Section II for details). GG has an Euclidean spanning ratio of $\Theta(\sqrt{n})$ [7], whereas PDT has a constant spanning ratio of at most $\frac{1+\sqrt{5}}{4}\pi^2 \approx 7.98$ [4]. Both PDT and GG can be constructed locally under the unit disk model, given only one-hop neighborhood information. There are additional subgraph topologies, for details see [5] and [10], and references therein.

B. Reactive topology control and greedy recovery

Guaranteed delivery beaconless forwarding (GDBF) [11] and *Beaconless forwarder planarization* (BFP) [5] are contention-based protocols for reactive computation of the GG neighborhood of a node. Both follow the *Select-and-Protest*-principle: In the *selection phase*, node u broadcasts a request and neighbors reply with a delay proportional to their distance to u. A node v possibly overhears other replies and may cancel its timer if such messages give evidence for (u, v) violating the GG rule. As it may occur that non-GG neighbors reply, an additional *protest phase* is required, in which protest messages correcting the outcome of the selection phase, can be sent. Since (possibly) not all neighbors reply, these protocols can be considered reactive. Executed on a node these protocols yield a local view on all incident Gabriel edges among which a localized routing algorithm can select the next hop for forwarding.

In fact, for greedy recovery it suffices to compute a single forwarding edge, provided it is free of edge intersections, rather than computing a node's entire subgraph neighborhood. There are numerous contention-based strategies that solve this problem in a non-reactive manner (see [8], [5], and references therein). To the best of our knowledge, the only two reactive protocols known are *Angular Relaying* (AR) [5] and the *Rotational Sweep* algorithm for boundary traversal (RS) [9].

AR is also a Select-and-Protest-based approach: First, a potential forwarding edge is selected according to an angle-based contention mechanism. In the protest phase, nodes may protest against this choice if it violates predefined planarity conditions (e.g., the GG rule). This process is repeated until a valid subgraph edge has been selected. However, the selected edges belong to planar non-spanners and in the worst-case, the protocol produces an unbounded message overhead.

In the contention-based RS algorithm, the timer's expiration can be best explained by the underlying geometric concept (see Fig. 4 in [9]): A sweep curve hinged at the executing node is rotated counter-clockwise. Once a node is hit by the sweep curve, it is selected for message forwarding. In contrast to AR, this algorithm uses a minimal amount of messages as it selects the next forwarding edge and transmits the message using only three messages (RTS-CTS-DATA). The selected edge is guaranteed to be intersection-free and belongs to a supergraph of the GG. Thus, the recovery path resulting from repeated execution of this protocol is shorter than or equally of the same length as the GG recovery path. However, it is not known if such a routing path is equally short as the corresponding PDT recovery path.

C. Our contributions

Extending the timer-based contention principle from [5], we introduce the first reactive algorithm for planar Euclidean spanner construction under the unit disk model. Executed on a network node, this node will successively learn about all its neighbors in the partial Delaunay triangulation (PDT neighbors). The algorithm is a RTS-CTS-based protocol that does not require any additional input other than the geographic position of its executing node. In contrast to the Select-and-Protest-based approaches, our algorithm consists solely of a selection phase and timers may be re-adjusted during protocol execution. We prove that each PDT neighbor sends a single message and that non-PDT neighbors do not. Thus, our algorithm constructs the PDT neighborhood with an optimal number of messages. We complement the presentation of the algorithm by a rigorous and comprehensive correctness proof.

The content is organized as follows: In Section II we introduce important definitions and notations. Our main contributions, the algorithm and its correctness proof are provided in Section III. The paper closes with a conclusion that additionally discusses future research directions in Section IV.

II. Preliminaries, definitions and notations

A. Network graph

We consider network graphs over a finite set V of nodes, which are represented by distinct points in the Euclidean plane. The Euclidean distance between any two such nodes $u, v \in V$ is denoted by $\|uv\|$. We apply the standard assumptions that no three nodes in V are collinear and that no four nodes in V are cocircular. The nodes in V are connected by a *unit disk graph* $\mathrm{UDG}(V)$. Given a positive constant R, such a graph connects any two nodes $u, v \in V$ with an edge iff $\|uv\| \leq R$. For convenience, we let $R = 1$. Furthermore, we always assume $\mathrm{UDG}(V)$ to be connected. The one-hop neighborhood $\mathrm{N}(u) = \{v \in V : \|uv\| \leq R\}$ of a node u is the set of all nodes reachable from u in one hop, including u itself.

B. Elementary geometric constructions

For three non-collinear points $u, v, w \in \mathbb{R}^2$, let $\mathrm{H}^w(u, v)$ denote the *open half-plane* that is bounded by the straight line defined by u and v, and contains w.

By $\mathrm{C}(u, v, w)$ we denote the unique circle with u, v, w on its boundary and by d_{uvw} we denote this circle's diameter. The interior of this circle (i.e., the circle without its boundary) is denoted by $\mathrm{C}^{\varnothing}(u, v, w)$. The disk that has the line segment between u and v as its diameter is denoted by $\mathrm{Disk}(u, v)$.

Let $\angle uvw$ denote the internal angle of the intersecting line segments (u, v) and (v, w). If $w \in \mathrm{Disk}(u, v)$, we frequently refer to w as the *angle maximizing* node w.r.t. (u,v), iff $\angle uwv \geq \angle uzv$, for all $z \in \mathrm{Disk}(u,v)$, $z \neq u,v$.

The proofs for Lemma 1 & 2 can be found in [3, Lemma 1].

Lemma 1: Let $u,v,w \in V$ s.t. $w \in \mathrm{Disk}(u,v)$ and let $\alpha = \angle uwv$. It holds that $\frac{\|uv\|}{\sin(\alpha)} = d_{uvw}$.

Lemma 2: Let $u,v,w \in V$ s.t. $w \in \mathrm{Disk}(u,v)$ and let $\alpha = \angle uwv$. Then it holds that $\sin(\alpha) \geq \|uv\|$ iff $d_{uvw} \leq 1$.

Lemma 3: Let $w \in \mathrm{Disk}(u,v)$ be the angle maximizing node w.r.t. (u,v) and let $\hat{w} \in \mathrm{Disk}(u,v)$ be another node. It holds that $d_{uvw} \geq d_{uv\hat{w}}$.

Proof: Let $\alpha = \angle uwv$ and $\hat{\alpha} = \angle u\hat{w}v$. Because w is assumed to be angle maximizing, $\alpha \geq \hat{\alpha}$. This is equivalent to $\sin(\alpha) \leq \sin(\hat{\alpha})$ (note that $\frac{\pi}{2} \leq \alpha, \hat{\alpha} \leq \pi$). By Lemma 1, $d_{uvw} = \|uv\|/\sin(\alpha)$ and $d_{uv\hat{w}} = \|uv\|/\sin(\hat{\alpha})$ holds. This implies $d_{uv\hat{w}} = \|uv\|/\sin(\hat{\alpha}) \leq \|uv\|/\sin(\alpha) = d_{uvw}$. ∎

C. Proximity graphs and their equivalence

Definition 1 (Partial Delaunay triangulation [3]): The *partial Delaunay triangulation* of a node set V, denoted by $\mathrm{PDT}(V)$, is defined as follows:

Let $(u,v) \in \mathrm{UDG}(V)$. $(u,v) \in \mathrm{PDT}(V)$, if and only if $\mathrm{Disk}(u,v)$ does not contain any node from V or for the angle maximizing node $w \in \mathrm{Disk}(u,v)$ the following holds:

1) there is no node $x \in \mathrm{C}(u,v,w) \cap \overline{\mathrm{H}^w(u,v)}$, and
2) $\sin(\alpha) \geq \|uv\|$, where $\alpha = \angle uwv$.[1]

Next, we give an alternative definition of PDT, which replaces the concept of an angle maximizing node by a statement that holds for all nodes contained in the Gabriel circle. Subsequently, we will show the equivalence of these definitions.

Definition 2 (Generalized partial Delaunay triangulation): The *generalized partial Delaunay triangulation* of V, denoted by $\mathrm{GPDT}(V)$, is defined as follows:

Let $(u,v) \in \mathrm{UDG}(V)$. $(u,v) \in \mathrm{GPDT}(V)$, if and only if the following holds for all nodes $w \in \mathrm{Disk}(u,v)$:

1) there is no node $x \in \mathrm{C}(u,v,w) \cap \overline{\mathrm{H}^w(u,v)}$, and
2) $\sin(\alpha) \geq \|uv\|$, where $\alpha = \angle uwv$.

Let $(u,v) \in \mathrm{UDG}(V)$ s.t. there exist nodes $w \in \mathrm{Disk}(u,v)$ and $x \in \mathrm{C}(u,v,w) \cap \overline{\mathrm{H}^w(u,v)}$. In such a setting, we use to say that edge (u,v) violates the first PDT (GPDT) criterion with *angular node* w and *witness node* x. Analogously, we use to say that (u,v) violates the second PDT (GPDT) criterion with *angular node* w.

Theorem 1: Applied on any $\mathrm{UDG}(V)$, the definitions of PDT and GPDT yield equivalent subgraphs, i.e., for $u,v \in V$ it holds that $(u,v) \in \mathrm{GPDT}(V) \Leftrightarrow (u,v) \in \mathrm{PDT}(V)$.

Proof: "$\Rightarrow$": Consider any edge $(u,v) \in \mathrm{GPDT}(V)$. If $\mathrm{Disk}(u,v)$ does not contain any node from V, $(u,v) \in \mathrm{PDT}(V)$ holds by definition. Otherwise, the fact that the two GPDT criteria hold for all nodes contained in $\mathrm{Disk}(u,v)$ implies that they hold in particular for the angle maximizing node $w \in \mathrm{Disk}(u,v)$. Thus, $(u,v) \in \mathrm{PDT}(V)$ follows.

"$\Leftarrow$": We now show the remaining case by proving that $(u,v) \notin \mathrm{GPDT}(V) \Rightarrow (u,v) \notin \mathrm{PDT}(V)$. Let $(u,v) \in \mathrm{UDG}(V)$ be an arbitrary edge which is not contained in $\mathrm{GPDT}(V)$. Then for at least one node $\hat{w} \in \mathrm{Disk}(u,v)$, at least one GPDT criterion is violated.

If the first GPDT criterion is violated, there exists at least one node $x \in \mathrm{C}(u,v,\hat{w})$ that lies on the opposite side than $\hat{w}$ w.r.t. (u,v). Let $w \in \mathrm{Disk}(u,v)$ be the angle maximizing node w.r.t. (u,v). Then w and $\hat{w}$ are either on the same or on opposite sides w.r.t. (u,v).

Assume the former holds. Then, according to Lemma 3, $d_{uvw} \geq d_{uv\hat{w}}$, because w is assumed to be angle maximizing. Moreover, because w and $\hat{w}$ are on the same side w.r.t. (u,v), $x \in \mathrm{C}(u,v,\hat{w})$ implies that $x \in \mathrm{C}(u,v,w)$. Thus, the first PDT criterion is violated and $(u,v) \notin \mathrm{PDT}(V)$ holds.

Now assume that w and $\hat{w}$ are on opposite sides w.r.t. (u,v). Then, because $w, \hat{w} \in \mathrm{Disk}(u,v)$, it holds that $w \in \mathrm{C}(u,v,\hat{w})$ and $\hat{w} \in \mathrm{C}(u,v,w)$. Again, the first PDT criterion is violated and $(u,v) \notin \mathrm{PDT}(V)$ holds.

Consider the case that the second GPDT criterion is violated. Let $\hat{\alpha} := \angle u\hat{w}v$ and $\alpha := \angle uwv$. It holds that $\sin(\hat{\alpha}) < \|uv\|$. The application of Lemma 2 yields that $d_{uv\hat{w}} > 1$ and by Lemma 3 $d_{uvw} \geq d_{uv\hat{w}}$ holds. Thus, according to Lemma 1, $\|uv\|/\sin(\alpha) = d_{uvw} > 1$. This is equivalent to $\sin(\alpha) < \|uv\|$. Thus, edge (u,v) violates the second PDT criterion, which implies $(u,v) \notin \mathrm{PDT}(V)$. ∎

III. Reactive partial Delaunay triangulation

A. The algorithm

Each node $z \in V$ is assumed to have a timer $\mathrm{t}(z)$ with readjustable timeout $\mathrm{timeout}_{\mathrm{t}(z)}$, describing the point in time of the timer's expiration, relative to the timer's starting point. By $\mathrm{expire}_{\mathrm{t}(z)}$ we refer to the ultimate timeout of $\mathrm{t}(z)$. We use the latter in the proofs to compare timeouts of nodes. Value $\mathrm{t}_{\max} > 0$ is an arbitrary maximal timeout.

Let $u \in V$ be an arbitrary node. Executed on u, algorithm Reactive-PDT works as follows:

(1) Node u starts the timer $\mathrm{t}(u)$ with $\mathrm{timeout}_{\mathrm{t}(u)} := \mathrm{t}_{\max}$ and broadcasts an RTS message including its position.

(2) Upon receiving an RTS message from u, node v initializes the *set of known nodes* $\mathrm{S}(v) := \emptyset$, the *current maximal angle* $\alpha_{\max}(v) := \pi/2$, and starts a timer $\mathrm{t}(v)$ with timeout $\mathrm{timeout}_{\mathrm{t}(v)} := \|uv\| \cdot \mathrm{t}_{\max}$.

(3) Once the timer $\mathrm{t}(v)$ of a node v expires, v broadcasts a CTS message including its position.

(4) Upon overhearing a CTS message from a node z, node v adds z to its list of known neighbors (i.e., sets $\mathrm{S}(v) := \mathrm{S}(v) \cup \{z\}$) and checks whether (u,v) violates the GPDT criteria using the set of known nodes $\mathrm{S}(v)$. If a violation is detected, v cancels its timer $\mathrm{t}(v)$ and remains silent. Otherwise, if $\alpha > \alpha_{\max}(v)$, where $\alpha = \angle uzv$, v sets $\alpha_{\max}(v) := \alpha$ and readjusts the timeout of timer $\mathrm{t}(v)$ to $\mathrm{timeout}_{\mathrm{t}(v)} := \frac{\|uv\|}{\sin(\alpha)} \cdot \mathrm{t}_{\max}$ (i.e., sets the timeout proportional to the diameter of $\mathrm{C}(u,z,v)$).

(5) If node u receives a CTS message from a node v, it adds v to the list of its PDT-neighbors. Once timer $\mathrm{t}(u)$ expires, u's list of PDT-neighbors has been completed.

[1] Note that criterion 2) would be $\sin(\alpha) > \|uv\|$ according to the original definition in [3]. However, PDT remains planar with our definition.

B. Structure of the correctness proof

The key observation is that once the executing node u has broadcasted its RTS message, u's neighbors reply with a CTS message if and only if they are PDT-neighbors of u (Theorem 2). Proving that each PDT-neighbor actually replies is rather simple, however, proving that non-PDT-neighbors do not, requires a deeper analysis of possible worst-case situations. Recall that none of the nodes is aware of its neighborhood. Thus, a non-PDT-neighbor v of u does not reply with a CTS message, iff it overhears, one or more CTS messages, prior to its timer's expiration, which give evidence of v not being a PDT-neighbor of u. That is, either v overhears a message of *angular node* w, s.t. u, v, w violate the second GPDT criterion, or it overhears messages of *angular node* w and *witness node* x, s.t. u, v, w, x violate the first or both GPDT criteria. Therefore, such nodes' existence as well as their timers' early expirations have to be proven. Note that the angular and the witness node have to be PDT-nodes.

However, the following worst-case situation may occur (consider Fig. 2b for an illustration): Assume (u, v) is not a GPDT edge. The angle maximizing node w_i w.r.t. (u, v) may itself not be a PDT-neighbor of u and therefore remain silent during algorithm execution. The angle maximizing node w_{i+1} w.r.t. (u, w_i) may also not be a PDT-neighbor of u and therefore remain silent, and so forth.

Such situations and the existence of a suitable angular node for (u, v) are considered in Lemmata 6–9. The existence of a suitable witness node is proven in Lemmata 10–12. Lemma 13 then deals with timers' early expirations. In Lemma 4 and Lemma 5, geometric properties, which are used frequently throughout the other proofs, are proven.

C. Correctness proof

Lemma 4: Let $u, v, w \in V$ s.t. $w \in \mathrm{Disk}(u, v)$ and assume w is the angle maximizing node w.r.t. (u, v). Then, the interior of $\mathrm{C}(u, v, w) \cap \mathrm{H}^w(u, v)$ is empty of nodes from V.

Proof: For the purpose of contradiction, assume that there exists $z \in \mathrm{Disk}(u, v)$ and $z \in \mathrm{C}(u, v, w) \cap \mathrm{H}^w(u, v)$. But then $\angle uwv < \angle uzv$ which contradicts the assumption that w is angle maximizing w.r.t. (u, v). ■

Lemma 5: Let $u, v, w, y \in V$ s.t. $w \in \mathrm{Disk}(u, v)$ and $y \in \mathrm{Disk}(u, w)$ are the angle maximizing nodes w.r.t. (u, v) and (u, w), respectively. Then it holds that (i) $d_{uyv} \geq d_{uyw}$, and (ii) $\mathrm{C}(u, y, w) \cap \mathrm{H}^w(u, y) \subseteq \mathrm{C}(u, y, v) \cap \mathrm{H}^w(u, y)$.

Proof: W.l.o.g. we may assume that edge (u, w) is parallel to the y-axis and that the y-coordinate of w is strictly larger than the y-coordinate of u. Consider Fig. 1 for an illustration. Let $\mathcal{L}$ be the straight line that is orthogonal to edge (u, w) and that passes through w. We define the following areas: Let A be the open half-plane that is bounded by $\mathcal{L}$ and which does not contain node u, let $B := A \cap \mathrm{C}^{\varnothing}(u, y, w)$, and let $C := A \cap \overline{\mathrm{H}^w(u, y)}$. Moreover, let $\alpha = \angle uwv$ and $\delta = \angle uyv$.

With $w \in \mathrm{Disk}(u, v)$, $\alpha \geq \frac{\pi}{2}$ holds and thus, $v \in A$. Moreover, because w is assumed to be angle maximizing w.r.t. (u, v) it holds that $\alpha \geq \delta$.

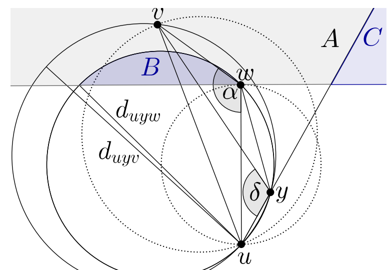


Fig. 1. Illustration of areas and node locations, used in the proof of Lemma 5

Assume $v \in B$. Let $\hat{v}$ be the point of intersection of the line segment between v and y with the boundary of $\mathrm{C}(u, y, w)$ in area A and denote the angle $\angle uw\hat{v}$ by $\hat{\alpha}$. By Lemma 1, $\|u\hat{v}\| / \sin(\hat{\alpha}) = d_{uyw} = \|u\hat{v}\| / \sin(\delta)$ and thus, $\hat{\alpha} = \delta$. Moreover, because $v \in B$ and $\hat{v} \notin B$, it follows $\|v\hat{v}\| > 0$. Therefore, $\alpha < \hat{\alpha} = \delta$ holds, which is a contradiction. Thus, $v \in A \setminus B$ holds and claim (i) follows directly.

Now, assume that $v \in C$. Then y is contained in the triangle defined by u, v, w (because of the definition of area C), which implies that $\delta > \alpha$. But this contradicts the assumption that w is angle maximizing w.r.t. (u, v) and thus, $v \notin C$. Along with the fact that $v \in A \setminus B$, claim (ii) follows. ■

Definition 3 (Hidden node sequence): Let (u, v) be any edge in $\mathrm{UDG}(V)$. By $\mathrm{HNS}(u, v) := \langle v = w_0, w_1, ..., w_k \rangle$, where $k \geq 0$, we denote a non-extendable sequence of nodes from V, s.t. the following holds for all nodes w_i, $0 \leq i \leq k$: $(u, w_i) \notin \mathrm{GPDT}(V)$ and $w_{i+1} \in \mathrm{Disk}(u, w_i)$ is angle maximizing w.r.t. (u, w_i).

We call such a sequence a *hidden node sequence* as these nodes will not send CTS messages during algorithm execution and will therefore remain hidden from v's local perspective.

Lemma 6: Let $(u, v) \in \mathrm{UDG}(V)$ be an arbitrary edge with $\mathrm{HNS}(u, v) = \langle v = w_0, w_1, ..., w_k = w \rangle$. Then there exists an angle maximizing node $y \in \mathrm{Disk}(u, w)$ w.r.t. (u, w), s.t. $(u, y) \in \mathrm{GPDT}(V)$.

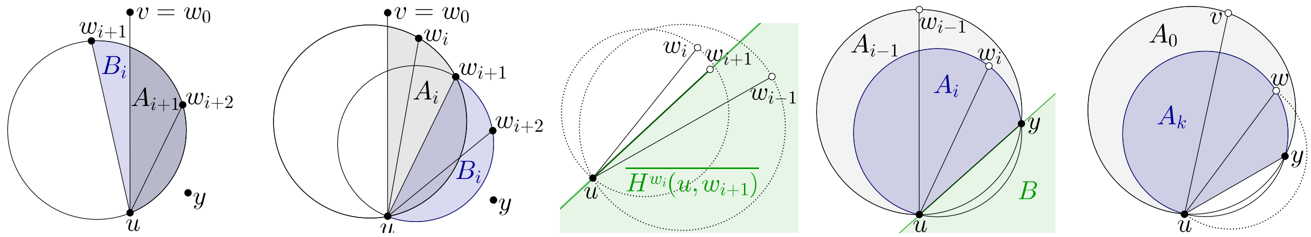
Proof: Because $(u, w) \notin \mathrm{GPDT}(V)$, there exists an angle maximizing node $y \in \mathrm{Disk}(u, w)$. Assume $(u, y) \notin \mathrm{GPDT}(V)$, then $\langle w_0, w_1, ..., w_k, y \rangle$ is a valid sequence of hidden nodes. But this contradicts to the assumption that the sequence is non-extendable and thus, the claim holds. ■

Lemma 7: Let $(u, v) \in \mathrm{UDG}(V)$ be an arbitrary edge. Let $\langle v = w_0, w_1, ..., w_k = w \rangle$, where $k \geq 0$, be a sequence of nodes s.t. $w_{i+1} \in \mathrm{Disk}(u, w_i)$ is angle maximizing w.r.t. (u, w_i). Let $y \in \mathrm{Disk}(u, w)$ be the angle maximizing node w.r.t. (u, w). Then $\mathrm{C}^{\varnothing}(u, y, w) \cap \mathrm{H}^y(u, v)$ is empty of nodes from V.

Proof: For simplicity, let $w_{k+1} := y$. Moreover, define the area $A_i := \mathrm{C}^{\varnothing}(u, w_{i+1}, w_i) \cap \mathrm{H}^y(u, v) \cap \mathrm{H}^{w_{i+1}}(u, v)$.

I.h.: For an arbitrary but fixed i it holds that area A_i is empty of nodes from V.

B.c. ($i = 0$): Area A_0 is defined as $A_0 = \mathrm{C}^{\varnothing}(u, w_1, w_0) \cap \mathrm{H}^y(u, v) \cap \mathrm{H}^{w_1}(u, v)$. Node w_1 is either contained in $\overline{\mathrm{H}^y(u, v)}$, or in $\mathrm{H}^y(u, v)$. Assume $w_1 \in \overline{\mathrm{H}^y(u, v)}$ holds. Then $\mathrm{H}^y(u, v) \cap \mathrm{H}^{w_1}(u, v) = \emptyset$. Thus, A_0 is empty of nodes from V. Now, let us assume $w_1 \in \mathrm{H}^y(u, v)$ holds. Then, w_1 and y lie within



(a) Case (i): $w_{i+1} \in \overline{\mathrm{H}^y(u,v)}$ (b) Case (ii): $w_{i+1} \in \overline{\mathrm{H}^y(u,v)}$ (c) $w_{i-1} \in \overline{\mathrm{H}^{w_i}(u,w_{i+1})}$ (d) I.s.: $A_i \subseteq A_{i-1}$ (e) Termination: $A_k \subseteq A_0$

Fig. 2. Illustrations for the proofs of Lemma 7 (a,b), Lemma 8 (c), and Lemma 9 (d,e)

the same half-plane w.r.t. (u,v) and it holds that $\mathrm{H}^y(u,v) = \mathrm{H}^{w_1}(u,v)$. Thus, A_0 simplifies to $A_0 = \mathrm{C}^{\varnothing}(u, w_1, w_0 = v) \cap \mathrm{H}^{w_1}(u,v)$. Because $w_1 \in \mathrm{Disk}(u,v)$ is assumed to be angle maximizing w.r.t. (u,v), Lemma 4 implies that A_0 is empty.

I.s. $(i \to i+1)$**:** Node w_{i+2} is either contained in $\overline{\mathrm{H}^y(u,v)}$, or in $\mathrm{H}^y(u,v)$. Assume $w_{i+2} \in \overline{\mathrm{H}^y(u,v)}$ holds. Then $\mathrm{H}^y(u,v) \cap \mathrm{H}^{w_{i+2}}(u,v) = \emptyset$. Thus, A_{i+1} is empty of nodes from V. Now, let us assume $w_{i+2} \in \mathrm{H}^y(u,v)$ holds. We distinguish the following two cases, which are illustrated in Fig. 2a and 2b, respectively: (i) $w_{i+1} \in \overline{\mathrm{H}^y(u,v)}$ and (ii) $w_{i+1} \in \mathrm{H}^y(u,v)$.

(i) Because $w_{i+2} \in \mathrm{Disk}(u, w_{i+1})$ is angle maximizing w.r.t. (u, w_{i+1}), Lemma 4 implies that area $B_i := \mathrm{C}(u, w_{i+2}, w_{i+1}) \cap \mathrm{H}^{w_{i+2}}(u, w_{i+1})$ is empty of nodes from V. Moreover, because $w_{i+2} \in \mathrm{H}^y(u,v)$ and $w_{i+1} \in \overline{\mathrm{H}^y(u,v)}$ holds, it follows that $A_{i+1} \subseteq B_i$ and thus, A_{i+1} is empty of nodes from V.

(ii) By i.h., area A_i is empty of nodes from V. Moreover, because w_{i+1} and y lie within the same half-plane w.r.t. (u,v) it holds that $\mathrm{H}^y(u,v) = \mathrm{H}^{w_{i+1}}(u,v)$. Thus, A_i simplifies to $A_i = \mathrm{C}^{\varnothing}(u, w_{i+1}, w_i) \cap \mathrm{H}^{w_{i+1}}(u,v)$. The fact that A_i is empty of nodes from V implies in particular that $w_{i+2} \notin A_i$. Thus, $\mathrm{C}(u, w_{i+2}, w_{i+1}) \cap \overline{\mathrm{H}^{w_{i+2}}(u, w_{i+1})} \cap \mathrm{H}^y(u,v) \subseteq A_i$ holds. Moreover, by Lemma 4 it holds that area $B_i := \mathrm{C}(u, w_{i+2}, w_{i+1}) \cap \mathrm{H}^{w_{i+2}}(u, w_{i+1})$ is empty of nodes from V. From the two latter observations we can conclude that area $A_{i+1} \subseteq A_i \cup B_i$ is also empty of nodes from V.

Termination $(i = k)$**:** The i.h. implies that the area $A_k = \mathrm{C}^{\varnothing}(u, w_{k+1}, w_k) \cap \mathrm{H}^y(u,v) \cap \mathrm{H}^{w_{k+1}}(u,v)$ is empty of nodes from V. With $w_{k+1} = y$ and $w_k = w$ it follows that $\mathrm{C}^{\varnothing}(u, y, w) \cap \mathrm{H}^y(u,v)$ is empty of nodes from V. ■

Lemma 8: Let $(u,v) \in \mathrm{UDG}(V)$ be an arbitrary edge with $\mathrm{HNS}(u,v) = \langle v = w_0, w_1, ..., w_k = w \rangle$. Let $y \in \mathrm{Disk}(u,w)$ be the angle maximizing node w.r.t. (u,w). Then it holds that $w_k, ..., w_1, w_0 \in \mathrm{H}^w(u,y)$.

Proof: First, notice the existence of such a node y according to Lemma 6, which additionally implies that $(u,y) \in \mathrm{GPDT}(V)$. We now show that all nodes w_i are located on the same side w.r.t. the straight line defined by u and y.

I.h.: For fixed i it holds that $w_k, ..., w_i \in \mathrm{H}^w(u,y)$.

B.c. $(i = k)$**:** $w_k = w \in \mathrm{H}^w(u,y)$ trivially holds.

I.s. $(i \to i-1)$**:** Assume $w_{i-1} \in \overline{\mathrm{H}^{w_i}(u, w_{i+1})}$ as illustrated in Fig. 2c. Because $w_{i+1} \in \mathrm{Disk}(u, w_i)$, $w_i \in \mathrm{Disk}(u, w_{i-1})$, and $w_i \in \mathrm{H}^{w_i}(u, w_{i+1})$ holds, it follows that $w_{i+1} \in \mathrm{Disk}(u, w_{i-1})$. Taking additionally into account that $w_{i-1} \in \overline{\mathrm{H}^{w_i}(u, w_{i+1})}$, it follows that $\angle u w_{i+1} w_{i-1} > \angle u w_i w_{i-1}$, which contradicts to w_i being angle maximizing w.r.t. (u, w_{i-1}). Hence, $w_{i-1} \in \mathrm{H}^{w_i}(u, w_{i+1})$ holds. Moreover, as $w_{i+1} \in \mathrm{H}^w(u,y)$ and $w_i \in \mathrm{H}^w(u,y)$ holds according to the i.h., it follows that $w_{i-1} \in \mathrm{H}^w(u,y)$. ■

Lemma 9: Let $(u,v) \in \mathrm{UDG}(V)$ be an arbitrary edge with $\mathrm{HNS}(u,v) = \langle v = w_0, w_1, ..., w_k = w \rangle$. Let $y \in \mathrm{Disk}(u,w)$ be angle maximizing w.r.t. (u,w). Then, (i) $d_{uyv} \geq d_{uyw}$ and (ii) $\mathrm{C}(u,y,w) \cap \mathrm{H}^v(u,y) \subseteq \mathrm{C}(u,y,v) \cap \mathrm{H}^v(u,y)$.

Proof: First, notice the existence of such a node y according to Lemma 6, which additionally implies that $(u,y) \in \mathrm{GPDT}(V)$. For simplicity, let $w_{k+1} := y$. Moreover, define the area $A_i := \mathrm{C}(u, y, w_i) \cap \mathrm{H}^v(u,y)$.

I.h.: For fixed i it holds that $A_k \subseteq A_{k-1} \subseteq \cdots \subseteq A_i$.

B.c. $(i = k)$**:** Because w_k and $w_{k+1} = y$ are the angle maximizing nodes w.r.t. (u, w_{k-1}) and (u, w_k), respectively, it holds by Lemma 5 that $\mathrm{C}(u,y,w_k) \cap \mathrm{H}^w(u,y) \subseteq \mathrm{C}(u,y,w_{k-1}) \cap \mathrm{H}^w(u,y)$. By Lemma 8 it holds that $v \in \mathrm{H}^w(u,y)$, which implies that $v \in \mathrm{H}^w(u,y) = \mathrm{H}^v(u,y)$. Thus, $\mathrm{C}(u,y,w_k) \cap \mathrm{H}^v(u,y) \subseteq \mathrm{C}(u,y,w_{k-1}) \cap \mathrm{H}^v(u,y)$, which is equivalent to $A_k \subseteq A_{k-1}$.

I.s. $(i \to i-1)$**:** Define $A_i^{\varnothing} := \mathrm{C}^{\varnothing}(u,y,w_i) \cap \mathrm{H}^v(u,y)$ and assume $w_{i-1} \in A_i^{\varnothing}$. Then, following the same line of argumentation as in Lemma 5, $\angle u y w_{i-1} > \angle u w_i w_{i-1}$. But this contradicts to the assumption that w_i is angle maximizing w.r.t. (u, w_{i-1}) and therefore, $w_{i-1} \in \overline{A_i^{\varnothing}}$ must hold (see Fig. 2d). Furthermore, Lemma 8 implies that $w_{i-1} \notin B := \overline{\mathrm{H}^v(u,y)}$. Thus, $w_{i-1} \in \overline{A_i^{\varnothing}} \cap \overline{B}$, which implies that $A_i \subseteq A_{i-1}$. The application of the i.h. yields $A_k \subseteq A_{k-1} \subseteq \cdots \subseteq A_i \subseteq A_{i-1}$.

Termination $(i = 0)$**:** With $A_k \subseteq A_0$ (from the i.h.), $w_k = w$, and $w_0 = v$ it follows that $\mathrm{C}(u,y,w) \cap \mathrm{H}^v(u,y) \subseteq \mathrm{C}(u,y,v) \cap \mathrm{H}^v(u,y)$. Thus, claim (ii) holds. Now consider claim (i): Because $y \in \mathrm{Disk}(u,w)$ and $w_{i+1} \in \mathrm{Disk}(u, w_i)$, for all $0 \leq i < k+1$, $\|uy\| = \|uw_{k+1}\| \leq \|uw_k\| \leq \cdots \leq \|uw_0\| = \|uv\|$ follows. Thus, the diameter of $\mathrm{C}(u,y,w)$ within $\mathrm{H}^y(u,w)$ is upper bounded by $\|uv\| \leq d_{uyv}$ (see Fig. 2e). Moreover, $A_k \subseteq A_0$ implies that the diameter of $\mathrm{C}(u,y,w)$ within $\overline{\mathrm{H}^y(u,w)}$ is at most d_{uyv}. The latter two statements imply that $d_{uyv} \geq d_{uyw}$. Thus, claim (i) holds. ■

Lemma 10: Let $u, v, w \in V$ be arbitrary nodes s.t. the diameter of $\mathrm{C}(u,v,w)$ is at most one, i.e., $d_{uvw} \leq 1$. If there exists a node $x \in \mathrm{C}(u,v,w)$ with $(u,x) \notin \mathrm{GPDT}(V)$, then there exists a node $\hat{x} \in \mathrm{C}(u,v,w)$ with $(u,\hat{x}) \in \mathrm{GPDT}(V)$.

Proof: Let $\mathrm{HNS}(u,x) = \langle x = y_0, y_1, ..., y_k = y \rangle$ be the hidden node sequence w.r.t. (u,x). According to Lemma 6, there exists an angle maximizing node $\hat{y} \in \mathrm{Disk}(u,y)$ w.r.t.

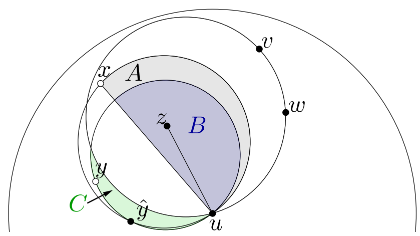
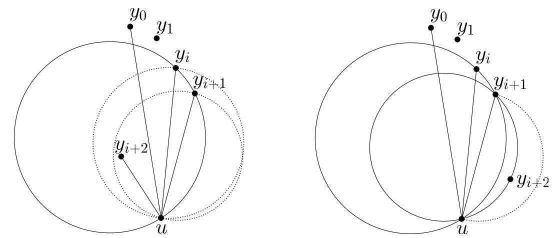


Fig. 3. Illustration of areas required for the proof of Lemma 10



(a) $y_{i+2} \in \mathrm{H}^{y_0}(u, y_{i+1})$ (b) $y_{i+2} \in \overline{\mathrm{H}^{y_0}(u, y_{i+1})}$

Fig. 4. Illustrations for the proof of Lemma 11

(u, y), s.t. $(u, \hat{y}) \in \mathrm{GPDT}(V)$. If $\hat{y} \in \mathrm{C}(u, w, v)$, then the claim holds with $\hat{x} := \hat{y}$. Thus, we assume $\hat{y} \notin \mathrm{C}(u, w, v)$ (see Fig. 3).

For later reference, we show first that $B := \mathrm{C}(u, \hat{y}, y) \cap \overline{\mathrm{H}^{\hat{y}}(u, x)} \subseteq \mathrm{C}(u, w, v)$. By Lemma 9 it holds that $\mathrm{C}(u, \hat{y}, y) \cap \mathrm{H}^{x}(u, \hat{y}) \subseteq \mathrm{C}(u, \hat{y}, x) \cap \overline{\mathrm{H}^{x}(u, \hat{y})}$. This in particular implies $B \subseteq A := \mathrm{C}(u, \hat{y}, x) \cap \overline{\mathrm{H}^{\hat{y}}(u, x)}$. Now assume there exists some point $p \in \mathbb{R}^2$, s.t. $p \in A$ and $p \notin \mathrm{Disk}(u, w, v)$. Then $\mathrm{C}(u, w, v)$ and $\mathrm{C}(u, \hat{y}, x)$ intersect four times, since $\hat{y}, p \notin \mathrm{C}(u, w, v)$, $x \in \mathrm{C}(u, w, v)$, but $\hat{y}$ and p are located on different sides of the straight line defined by u and x. This contradicts elementary geometry and therefore cannot occur. Thus, $A \subseteq \mathrm{C}(u, w, v)$ which finally implies $B \subseteq A \subseteq \mathrm{C}(u, w, v)$

Observe that (u, y) cannot violate the second PDT criterion, as $B \subseteq \mathrm{C}(u, v, w)$ and $d_{uvw} \leq 1$, implying that $d_{u\hat{y}y} \leq 1$ (see Lemma 2). Moreover, because $(u, y) \notin \mathrm{GPDT}(V)$, (u, y) must violate the first PDT criterion. Thus, there exists a node $z \in \mathrm{C}(u, \hat{y}, y)$. In accordance with Lemma 7, there cannot exist a node within $\mathrm{C}^{\varnothing}(u, \hat{y}, y) \cap \mathrm{H}^{\hat{y}}(u, x)$ and therefore, $z \in B$ and in particular $z \in \mathrm{C}(u, w, v)$ must hold.

If $(u, z) \in \mathrm{GPDT}(V)$, our claim holds with $\hat{x} := z$. Otherwise, we can repeat the arguments used above inductively with z instead of x and circle $\mathrm{C}(u, \hat{y}, y)$ instead of circle $\mathrm{C}(u, v, w)$. Note that each node that is contained by $\mathrm{C}(u, \hat{y}, y)$ must also be contained by $\mathrm{C}(u, w, v)$, because Lemma 6 particularly implies that area $C := \overline{\mathrm{C}(u, v, w)} \cap \mathrm{C}(u, \hat{y}, y)$ is empty of nodes from V. Furthermore, note that this induction eventually terminates, because $x \notin \mathrm{C}(u, \hat{y}, y)$ (which is due to the fact that $B \subseteq A$ and the assumption that no four points in V are cocircular) cannot take over the role of z in the induction step. This implies in particular that the number of nodes (candidates), which could possibly play z's role, shrinks by at least one in the induction step. Now assume there would be no candidate left after the induction's termination. Then, (u, y) would neither violate the first, nor the second PDT criterion, which would imply that $(u, y) \in \mathrm{GPDT}(V)$. But this would contradict the initial assumption that $\mathrm{HNS}(u, v) = \langle v = y_0, y_1, ..., y_k = y \rangle$ is a hidden node sequence. That is, the only possibility of the induction's termination without finding a GPDT-neighbor of u within $\mathrm{C}(u, w, v)$ results in a contradiction. From this we conclude that such a GPDT-neighbor of u must exist in $\mathrm{C}(u, w, v)$. ■

Lemma 11: Let $u \in V$ be an arbitrary node and let $\langle y_0, y_1, ..., y_k \rangle$ be a sequence of nodes from V, s.t. $(u, y_0) \in \mathrm{GPDT}(V)$ holds and $y_{i+1} \in \mathrm{Disk}(u, y_i)$ is angle maximizing w.r.t. (u, y_i). Then, $y_k \in \mathrm{GPDT}(V)$ holds.

Proof: The main observation in this proof is that all edges (u, y_i) are contained in $\mathrm{GPDT}(V)$. We prove this by induction on the index i, where the induction hypothesis is tripartite: (ii) and (iii) correspond to the first and the second GPDT criterion, respectively, whereas (i) just serves to prove the other two. It reflects that all nodes y_i are located in the same half-plane w.r.t. (u, y_0) (see Fig. 4b).

I.h.: For an arbitrary but fixed i, the following holds:

(i) $y_{j+1} \in \overline{\mathrm{H}^{y_0}(u, y_j)}$, for all $1 \leq j \leq i$

(ii) $\mathrm{C}(u, y_1, y_0) \cap \overline{\mathrm{H}^{y_1}(u, y_0)} \supseteq \ldots \supseteq \mathrm{C}(u, y_{i+1}, y_i) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$

(iii) $d_{uy_1y_0} \geq d_{uy_2y_1} \geq \ldots \geq d_{uy_{i+1}y_i}$

B.c. ($i = 1$): From Lemma 4 and the assumption that $(u, y_0) \in \mathrm{GPDT}(V)$ it immediately follows that $\mathrm{C}^{\varnothing}(u, y_1, y_0)$ is empty of nodes from V. Now, assume to the contrary of i.h. (i) that $y_2 \in \mathrm{H}^{y_0}(u, y_1)$ (see Fig. 4a with $i = 0$). Since $y_1 \in \mathrm{Disk}(u, y_0)$, it follows that $\mathrm{Disk}(u, y_1) \cap \mathrm{H}^{y_2}(u, y_0) \subseteq \mathrm{C}^{\varnothing}(u, y_1, y_0)$, which implies the emptiness of nodes from V as this holds for $\mathrm{C}^{\varnothing}(u, y_1, y_0)$. According to Lemma 4, area $\mathrm{Disk}(u, y_1) \cap \mathrm{H}^{y_2}(u, y_0)$ can be extended to $\mathrm{Disk}(u, y_1) \cap \mathrm{H}^{y_2}(u, y_1) = \mathrm{Disk}(u, y_1) \cap \mathrm{H}^{y_0}(u, y_1)$ and remains empty of nodes from V. This contradicts the assumption that $y_2 \in \mathrm{Disk}(u, y_1) \cap \mathrm{H}^{y_0}(u, y_1)$ and thus, part (i) of the i.h. holds.

Moreover, because $y_2 \in \overline{\mathrm{H}^{y_0}(u, y_1)}$ and $y_1 \in \mathrm{Disk}(u, y_0)$, it follows that $\mathrm{C}(u, y_1, y_0) \cap \overline{\mathrm{H}^{y_1}(u, y_0)} \supseteq \mathrm{C}(u, y_2, y_1) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$ (see Fig. 4b with $i = 0$), which in turn implies that $d_{uy_1y_0} \geq d_{uy_2y_1}$. We conclude that part (ii) and (iii) of the i.h. hold for $i = 1$.

I.s. ($i \rightarrow i + 1$): By assumption $(u, y_0) \in \mathrm{GPDT}(V)$ and therefore, $\mathrm{C}(u, y_1, y_0) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$ is empty of nodes from V. With part (ii) of the i.h. this also holds for $\mathrm{C}(u, y_{i+1}, y_i) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$. Furthermore, Lemma 7 implies that $\mathrm{C}^{\varnothing}(u, y_{i+1}, y_i) \cap \mathrm{H}^{y_{i+1}}(u, y_0)$ is empty of nodes from V, which is equivalent to $\mathrm{C}^{\varnothing}(u, y_{i+1}, y_i) \cap \mathrm{H}^{y_1}(u, y_0)$ being empty, because y_1 and y_{i+1} are on the same side w.r.t. the straight line defined by u and y_0, according to i.h. (i). From this it follows that $\mathrm{C}^{\varnothing}(u, y_{i+1}, y_i)$ is empty of nodes.

Now, assume to the contrary of i.h. (i) that $y_{i+2} \in \mathrm{H}^{y_0}(u, y_{i+1})$ (see Fig. 4a). According to i.h. (i), it holds that $y_{i+1} \in \overline{\mathrm{H}^{y_0}(u, y_i)}$. Because additionally $y_{i+1} \in \mathrm{Disk}(u, y_i)$ holds, it follows that $\mathrm{Disk}(u, y_{i+1}) \cap \mathrm{H}^{y_0}(u, y_i) \subseteq \mathrm{Disk}(u, y_i) \cap \mathrm{H}^{y_0}(u, y_i) \subseteq \mathrm{C}^{\varnothing}(u, y_{i+1}, y_i)$. As we already proved $\mathrm{C}^{\varnothing}(u, y_{i+1}, y_i)$ to be empty of nodes from V, this holds in particular for $\mathrm{Disk}(u, y_{i+1}) \cap \mathrm{H}^{y_0}(u, y_i)$. According to Lemma 7, area $\mathrm{Disk}(u, y_{i+1}) \cap \mathrm{H}^{y_0}(u, y_i)$ can be extended to $\mathrm{Disk}(u, y_{i+1}) \cap \mathrm{H}^{y_0}(u, y_{i+1})$ and remains empty of nodes from V, which contradicts the assumptions that $y_{i+2} \in \mathrm{Disk}(u, y_{i+1}) \cap \mathrm{H}^{y_0}(u, y_{i+1})$ and thus, part (i) of the i.h. holds for $i + 1$.

Moreover, because $y_{i+1} \in \overline{\mathrm{H}^{y_0}(u, y_i)}$, $y_{i+1} \in \overline{\mathrm{H}^{y_0}(u, y_i)}$ and $y_{i+1} \in \mathrm{Disk}(u, y_i)$ hold, it follows that $\mathrm{C}(u, y_{i+1}, y_i) \cap \overline{\mathrm{H}^{y_1}(u, y_0)} \supseteq \mathrm{C}(u, y_{i+2}, y_{i+1}) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$ (see Fig. 4b). The latter together with $y_{i+2} \in \mathrm{Disk}(u, y_{i+1})$ implies $d_{uy_{i+1}y_i} \geq d_{uy_{i+2}y_{i+1}}$. Thus, parts (ii) and (iii) of the i.h. hold for $i + 1$.

Termination: Considering (u, y_k), two cases may occur: either there exists an angle maximizing node $y_{k+1} \in \mathrm{Disk}(u, y_k)$, or not. In the latter case $(u, y_k) \in \mathrm{GPDT}(V)$ holds trivially. Therefore, let us assume the former holds. Because $(u, y_0) \in \mathrm{GPDT}(V)$, it holds that $\mathrm{Disk}(u, y_1, y_0) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$ is empty of nodes from V. Applying i.h. (ii) yields that $\mathrm{Disk}(u, y_{k+1}, y_k) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$ is also empty of nodes from V. Now, from i.h. (i) it follows that $y_2 \in \overline{\mathrm{H}^{y_0}(u, y_1)}$, $y_3 \in \overline{\mathrm{H}^{y_0}(u, y_2)}$, up to $y_{k+1} \in \overline{\mathrm{H}^{y_0}(u, y_k)}$. This implies that all nodes $y_1, y_2, y_3, \ldots, y_{k+1}$ are located in the same half-plane w.r.t. (u, y_0). Because especially y_1 and y_{k+1} are in the same half-plane, it follows that $\mathrm{C}(u, y_{k+1}, y_k) \cap \overline{\mathrm{H}^{y_{k+1}}(u, y_0)} = \mathrm{C}(u, y_{k+1}, y_k) \cap \overline{\mathrm{H}^{y_1}(u, y_0)}$ holds. According to Lemma 7, it follows that the empty area $\mathrm{C}(u, y_{k+1}, y_k) \cap \overline{\mathrm{H}^{y_{k+1}}(u, y_0)}$ can be extended to $\mathrm{C}(u, y_{k+1}, y_k) \cap \overline{\mathrm{H}^{y_{k+1}}(u, y_k)}$ and is still empty of nodes from V. Thus, (u, y_k) cannot violate the first GPDT criterion. According to i.h. (iii) and the fact that $(u, y_0) \in \mathrm{GPDT}(V)$, it follows that $d_{uy_{k+1}y_k} \leq d_{uy_1y_0} \leq 1$. Thus, (u, y_k) is also not violating the second GPDT criterion and therefore, $(u, y_k) \in \mathrm{GPDT}(V)$ holds. ■

Lemma 12: Given four nodes $u, v, w, x \in V$, s.t. $d_{uvw} \leq 1$ and $x \in \mathrm{C}(u, v, w)$, then there exists a node $\hat{x} \in V$ satisfying the following three properties: (i) $(u, \hat{x}) \in \mathrm{GPDT}(V)$, (ii) $\hat{x} \in \mathrm{C}(u, v, w)$, and (iii) $d_{u\hat{y}\hat{x}} \leq d_{uvw}$, if there exists an angle maximizing node $\hat{y} \in \mathrm{Disk}(u, \hat{x})$ w.r.t. $(u, \hat{x})$.

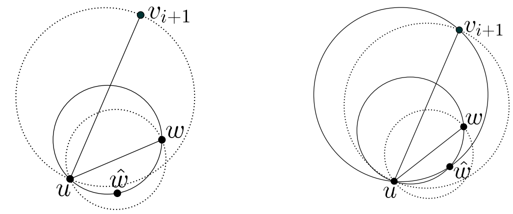
Proof: Edge (u, x) may, or may not be contained in $\mathrm{GPDT}(V)$. If $(u, x) \notin \mathrm{GPDT}(V)$, then replace x by the node x' whose existence is guaranteed by Lemma 10 and for which it additionally holds by the Lemma that $x' \in \mathrm{C}(u, v, w)$ and $(u, x') \in \mathrm{GPDT}(V)$.

Consider the longest sequence $S := \langle x = y_0, y_1, ..., y_k \rangle$ s.t. $y_{i+1} \in \mathrm{Disk}(u, y_i)$ is angle maximizing w.r.t. (u, y_i) and $y_k \in \mathrm{C}(u, v, w)$ holds. Apply Lemma 11 on the sequence S and observe that $(u, y_k) \in \mathrm{GPDT}(V)$ holds. With $\hat{x} := y_k$, part (i) and part (ii) of the claim immediately hold.

Now, let us assume there exists an angle maximizing node $\hat{y} \in \mathrm{Disk}(u, \hat{x})$ w.r.t. $(u, \hat{x})$ (otherwise, we are done). Then $\hat{y} \notin \mathrm{C}(u, v, w)$ must hold, because otherwise $S \cup \{\hat{y}\}$ would be a valid sequence longer than S, contradicting the maximality of S. Finally, since $\hat{y} \notin \mathrm{C}(u, v, w)$, $\hat{y} \in \mathrm{Disk}(u, \hat{x})$, and $\hat{x} \in \mathrm{C}(u, v, w)$, it follows that $d_{u\hat{y}\hat{x}} \leq d_{uvw}$. ■

Lemma 13: Let $(u, v) \in \mathrm{UDG}(V)$ be an arbitrary edge and let $w \in \mathrm{Disk}(u, v)$ be the angle maximizing GPDT-node w.r.t. (u, v), i.e. $(u, w) \in \mathrm{GPDT}(V)$. It holds that timer $\mathrm{t}(w)$ expires earlier than timer $\mathrm{t}(v)$, i.e. $\mathrm{expire}_{\mathrm{t}(w)} < \mathrm{expire}_{\mathrm{t}(v)}$.

Proof: Let sequence $\langle v_1, v_2, ..., v_k \rangle$ represent the ordered neighborhood of u, where $\|uv_i\| \leq \|uv_{i+1}\|$, for all $1 \leq i < k$.



(a) Case: $\hat{w} \notin \mathrm{Disk}(u, v_{i+1})$ (b) Case: $\hat{w} \in \mathrm{Disk}(u, v_{i+1})$

Fig. 5. Illustrations for the proof of Lemma 13

W.l.o.g. we assume $\mathrm{t}_{\max} = 1$.

I.h. Let i be arbitrary but fixed. For the angle maximizing GPDT-node $w \in \mathrm{Disk}(u, v_j)$ it holds that $\mathrm{expire}_{\mathrm{t}(w)} < \mathrm{expire}_{\mathrm{t}(v_j)}$, for all $1 \leq j \leq i$.

B.c. ($i = 1$): The i.h. for $i = 1$ holds trivially, because there is no node closer to u than v_1.

I.s. ($i \rightarrow i+1$): Let $w \in \mathrm{Disk}(u, v_{i+1})$ be the angle maximizing GPDT-node w.r.t. (u, v_{i+1}).

If $\mathrm{Disk}(u, w)$ does not contain any GPDT-node, then according to the algorithm $\mathrm{expire}_{\mathrm{t}(w)} = \|uw\|$. Moreover, $\mathrm{expire}_{\mathrm{t}(v_{i+1})} \geq \|uv_{i+1}\|$ and because $w \in \mathrm{Disk}(u, v_{i+1})$, it follows that $\mathrm{expire}_{\mathrm{t}(v_{i+1})} \geq \|uv_{i+1}\| > \|uw\| = \mathrm{expire}_{\mathrm{t}(w)}$. Thus, the i.h. for $i + 1$ holds.

For the remainder, assume the existence of an angle maximizing GPDT-node $\hat{w} \in \mathrm{Disk}(u, w)$ w.r.t. (u, w).

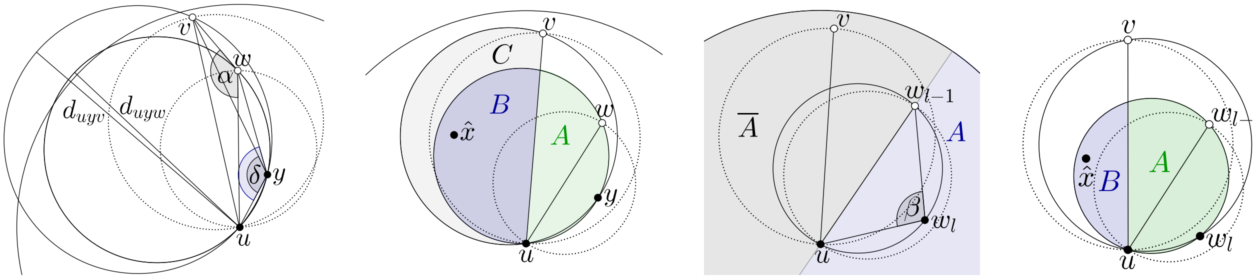
Assume $\hat{w} \notin \mathrm{Disk}(u, v_{i+1})$. Then, because it additionally holds that $w \in \mathrm{Disk}(u, v_{i+1})$ and $\hat{w} \in \mathrm{Disk}(u, w)$, it follows that $\|uv_{i+1}\| > d_{u\hat{w}w}$ (see Fig. 5a). As shown above, it holds that $\|uv_{i+1}\| > \|uw\|$. Because $\hat{w}$ is the angle maximizing GPDT-node w.r.t. (u, w), during the algorithm's execution the timeout of timer $\mathrm{t}(w)$ can be increased to at most $d_{u\hat{w}w}$. Thus, $\mathrm{expire}_{\mathrm{t}(w)} \leq \max\{d_{u\hat{w}w}, \|uw\|\}$. Hence, $\mathrm{expire}_{\mathrm{t}(v_{i+1})} \geq \|uv_{i+1}\| > \max\{d_{u\hat{w}w}, \|uw\|\} \geq \mathrm{expire}_{\mathrm{t}(w)}$ and thus, the i.h. holds for $i + 1$.

Now, assume $\hat{w} \in \mathrm{Disk}(u, v_{i+1})$ (see Fig. 5b). Because $\|uw\| < \|uv_{i+1}\|$ and $\hat{w}$ is the angle maximizing GPDT-node w.r.t. (u, w), the i.h. implies that $\mathrm{expire}_{\mathrm{t}(\hat{w})} < \mathrm{expire}_{\mathrm{t}(w)}$. Thus, according to the algorithm, $\hat{w}$ sends a CTS message before timer $\mathrm{t}(w)$ expires. Therefore, and because $\hat{w} \in \mathrm{Disk}(u, v_{i+1})$, during the algorithm's execution $\mathrm{timeout}_{\mathrm{t}(v_{i+1})}$ will be set to $d_{u\hat{w}v_{i+1}}$ (or is is already at least as large) before timer $\mathrm{t}(v_{i+1})$ expires. Now, because $w \in \mathrm{Disk}(u, v_{i+1})$ and $\hat{w} \in \mathrm{Disk}(u, w)$ are the angle maximizing GPDT-nodes w.r.t. (u, v_{i+1}) and (u, w), respectively, Lemma 5 implies that $d_{u\hat{w}w} < d_{u\hat{w}v_{i+1}}$ (where the strict inequality holds due to the non-cocircularity assumption). Therefore, it follows that $\mathrm{expire}_{\mathrm{t}(v_{i+1})} \geq d_{u\hat{w}v_{i+1}} > d_{u\hat{w}w} = \max\{d_{u\hat{w}w}, \|uw\|\} \geq \mathrm{expire}_{\mathrm{t}(w)}$. Thus, the i.h. holds for $i + 1$. ■

Theorem 2: Let $\mathrm{UDG}(V)$ be an arbitrary unit disk graph. Execute algorithm REACTIVE-PDT on an arbitrary node $u \in V$. Then, within time at most $\mathrm{t}_{\max}$, each neighbor $v \in \mathrm{N}(u)$ sends a CTS message, iff $(u, v) \in \mathrm{GPDT}(V)$.

Proof: Let $v \in \mathrm{N}(u)$ be an arbitrary node. In the following, we show separately: v sends a CTS message if $(u, v) \in \mathrm{GPDT}(V)$ and v does not send a CTS message if $(u, v) \notin \mathrm{GPDT}(V)$. W.l.o.g., we assume $\mathrm{t}_{\max} = 1$.

"$\Rightarrow$" Because $(u, v) \in \mathrm{GPDT}(V)$, during algorithm execu-



(a) Case 1.1 (b) Case 1.2 (c) Case 2.1 (d) Case 2.2

Fig. 6. Illustrations for the distinguished cases in the proof of Theorem 2

tion v will not detect any violation of a GPDT criterion. Thus, v's timer expires eventually and it sends a CTS message.

Consider the point in time when timer $\mathrm{t}(v)$ expires: It holds that $\mathrm{expire}_{\mathrm{t}(v)} = \max\{d_{uwv}, \|uv\|\}$, where $\|uv\|$ is the diameter of $\mathrm{Disk}(u,v)$ and where d_{uwv} is the diameter of $\mathrm{C}(u,w,v)$ with node w being the node, which maximizes d_{uwv} among all nodes contained in $\mathrm{Disk}(u,v)$ sending a CTS.

Assume $d_{uwv} > 1$ holds. Then, (u,v) violates the second GPDT criterion with angle maximizing node w, which contradicts to (u,v) being contained in $\mathrm{GPDT}(V)$. With $v \in \mathrm{N}(u)$, $\|uv\| \leq 1$ holds and thus, $\mathrm{expire}_{\mathrm{t}(v)} = \max\{d_{uwv}, \|uv\|\} \leq 1$, which proves the first part of the theorem.

"$\Leftarrow$" Assume $(u,v) \notin \mathrm{GPDT}(V)$. Let $\mathrm{HNS}(u,v) = \langle v = w_0, w_1, ..., w_k = w\rangle$ be the hidden node sequence w.r.t. (u,v). According to Lemma 6, there exists an angle maximizing node $y \in \mathrm{Disk}(u,w)$ w.r.t. (u,w) s.t. $(u,y) \in \mathrm{GPDT}(V)$ holds. If there are several such sequences, choose the sequence s.t. y is the angle maximizing node w.r.t. (u,v) (considering only nodes y for which $(u,y) \in \mathrm{GPDT}(V)$ holds). Node y may, or may not be contained in $\mathrm{Disk}(u,v)$. In the following, we will distinguish these two cases and prove them separately. For each case we need to show

1) the existence of nodes that witness the violation of at least one of the GPDT criteria for edge (u,v),
2) the visibility of the aforementioned nodes, i.e., these nodes are GPDT-nodes, and their timers expire earlier than the timer of v, and
3) that these nodes are neighbors of v in $\mathrm{UDG}(V)$.

The first property is needed for v being able to detect a violation of the GPDT criteria during algorithm execution. The second property ensures that these nodes send a CTS message timely. Finally, the third property ensures that v can actually overhear these CTS messages and therefore, these nodes are contained in v's set of known neighbors $\mathrm{S}(v)$.

Case 1: Assume $y \in \mathrm{Disk}(u,v)$. Node w might be hidden because (u,w) either violates the first, or the second PDT criterion (if both criteria are violated, consider the violation of the second criterion, handled in Case 1.1).

Case 1.1: Assume (u,w) violates the second PDT criterion. It holds that $\sin(\beta) < \|uw\|$, where $\beta = \angle uyw$ (see Fig. 6a). By Lemma 1, $d_{uyw} = \|uw\|/\sin(\beta)$ and $d_{uyv} = \|uv\|/\sin(\delta)$, where $\delta = \angle uyv$. Moreover, property (i) of Lemma 9 implies that $d_{uyv} \geq d_{uyw}$. Thus, $\|uv\|/\sin(\delta) = d_{uyv} \geq d_{uyw} = \|uw\|/\sin(\beta) > 1 \Leftrightarrow \sin(\delta) < \|uv\|$. That is, (u,v) violates the second GPDT criterion with angular node y. Because $y \in \mathrm{Disk}(u,v)$ is the angle maximizing GPDT-node w.r.t. (u,v) (by definition of $\mathrm{HNS}(u,v)$), Lemma 13 implies that timer $\mathrm{t}(y)$ expires earlier than timer $\mathrm{t}(v)$. Thus, y sends a CTS-message before timer $t(v)$ expires. Since v is contained within y's unit disk, y will be added to $S(v)$. Therefore, during algorithm execution, node v is able to detect a violation of the second GPDT criterion for (u,v) with angular node $y \in \mathrm{S}(v)$.

Case 1.2: Assume (u,w) violates the first PDT criterion. Let $A := \mathrm{C}(u,y,w) \cap \mathrm{H}^y(u,v)$, $B := \mathrm{C}(u,y,w) \cap \overline{\mathrm{H}^y(u,v)}$, and $C := \mathrm{C}(u,y,v) \cap \overline{\mathrm{H}^y(u,v)}$ (see Fig. 6b).

Because (u,w) violates the first PDT criterion with angle maximizing node y, there exists at least one node $x \in \mathrm{C}(u,y,w)$. The application of Lemma 12 on $\mathrm{C}(u,y,w)$ and x guarantees the existence of a node $\hat{x}$ with $(u,\hat{x}) \in \mathrm{GPDT}(V)$ and $d_{u\hat{y}\hat{x}} \leq d_{uyw}$ (if there exists an angle maximizing node $\hat{y} \in \mathrm{Disk}(u,\hat{x})$ w.r.t. $(u,\hat{x})$). Moreover, Lemma 7 implies that $\hat{x} \in B$, because there cannot exist any node within A. Additionally, because $B \subseteq C$ (Lemma 9), $\hat{x} \in C$ holds. This implies that (u,v) violates the first GPDT criterion with angular node y and witness $\hat{x}$. In addition, $d_{uyw} < d_{uyv}$ holds by Lemma 9 and the assumption that no four points in V are cocircular. If $\hat{y}$ exists, we can conclude that $\mathrm{expire}_{\mathrm{t}(\hat{x})} \leq d_{u\hat{y}\hat{x}} \leq d_{uyw} < d_{uyv} \leq \mathrm{expire}_{\mathrm{t}(v)}$ holds. If $\hat{y}$ does not exist, it follows that $\mathrm{expire}_{\mathrm{t}(\hat{x})} = \|u\hat{x}\| \leq d_{uyw} < d_{uyv} \leq \mathrm{expire}_{\mathrm{t}(v)}$. Thus, in either case timer $\mathrm{t}(\hat{x})$ expires earlier than timer $\mathrm{t}(v)$.

Moreover, because $y \in \mathrm{Disk}(u,v)$ and y is the angle maximizing GPDT-node w.r.t. (u,v), by Lemma 13 timer $\mathrm{t}(y)$ expires earlier than timer $\mathrm{t}(v)$, i.e., y sends a CTS message before timer $\mathrm{t}(v)$ expires. Furthermore, v is contained in y's unit disk and can overhear this message, from which it follows that $y \in \mathrm{S}(v)$ holds before v's timer expires. This also holds for $\hat{x}$ as it is a GPDT-node and its timer $\mathrm{t}(\hat{x})$ expires earlier than timer $\mathrm{t}(v)$. Moreover, $\hat{x} \in C \subseteq \mathrm{C}(u,y,v)$ with $d_{uyv} \leq 1$ and thus, it holds that v can overhear $\hat{x}$'s CTS message and $\hat{x} \in \mathrm{S}(v)$ holds before the timer of v expires. Thus, when v's timer expires, $\{\hat{x}, y\} \subseteq \mathrm{S}(v)$ holds and v can detect a violation of the first GPDT criterion with angular node y and witness node $\hat{x}$ and will therefore remain silent.

Case 2: Assume $y \notin \mathrm{Disk}(u,v)$. We show that this case cannot occur. For simplicity, let $w_{k+1} := y$. Let l be the smallest index w.r.t. $\mathrm{HNS}(u,v) \cup \{w_{k+1}\} = \langle w_0, w_1, \ldots, w_{k+1}\rangle$, s.t. w_l is located outside of $\mathrm{Disk}(u,v)$. Then node w_{l-1} is a hidden node, located in $\mathrm{Disk}(u, v)$. In the following case distinction we now prove that (u, w_{l-1}) can neither violate the first (Case 2.2), nor the second PDT criterion (Case 2.1), contradicting our initial assumption on $\mathrm{HNS}(u, v)$. Thus, all nodes w_i, $0 \leq i \leq k+1$, have to be located in $\mathrm{Disk}(u, v)$.

Case 2.1: Assume (u, w_l) violates the second PDT criterion. Recall that $w_{l-1} \in \mathrm{Disk}(u, v)$, whereas w_l is assumed to be strictly outside $\mathrm{Disk}(u, v)$. Therefore, $\mathrm{Disk}(u, v) \neq \mathrm{C}(u, w_{k+1}, w_{l-1})$ and these circles intersect exactly twice in $A := \mathrm{H}^{w_l}(u, w_{l-1})$ (see Fig. 6c). In order to violate the second PDT criterion, there must exists at least one point $p \in \mathbb{R}^2$, s.t. $p \in \mathrm{C}(u, w_l, w_{l-1}) \cap \overline{A}$ and p is not contained in $\mathrm{Disk}(u, v)$ (in fact, p must be outside of u's unit disk). But then, $\mathrm{Disk}(u, v)$ and $\mathrm{C}(u, w_l, w_{l-1})$ intersect in four points, which is a contradiction to elementary geometry. Thus, (u, w_{l-1}) cannot violate the second PDT criterion.

Case 2.2: Assume (u, w_l) violates the first PDT criterion. Let $A := \mathrm{C}(u, w_l, w_{l-1}) \cap \mathrm{H}^{w_l}(u, v)$ and let $B := \overline{\mathrm{C}(u, w_l, w_{l-1}) \cap \overline{\mathrm{H}^{w_l}(u, v)}}$ (see Fig. 6d). Because (u, w_{l-1}) violates the first PDT criterion with angle maximizing node w_l, there exists at least one node $x \in \mathrm{C}(u, w_l, w_{l-1})$. Applying Lemma 12 on $\mathrm{C}(u, w_l, w_{l-1})$ and x guarantees the existence of a node $\hat{x}$ with $(u, \hat{x}) \in \mathrm{GPDT}(V)$. Moreover, Lemma 7 implies that $\hat{x} \in B$, because there cannot exist any node in A. Next, we show by contradiction that $\hat{x} \in \mathrm{Disk}(u, v)$ holds.

Assume $\hat{x} \notin \mathrm{Disk}(u, v)$ and recall that w_l and $\hat{x}$ lie on different sides of the straight line defined by u and v. Because $w_l \notin \mathrm{Disk}(u, v)$ and $w_{l-1} \in \mathrm{Disk}(u, v)$, $\mathrm{Disk}(u, v)$ intersects $\mathrm{C}(u, w_l, w_{l-1})$ twice in $\mathrm{H}^{w_l}(u, v)$. Because $\hat{x} \in \overline{\mathrm{H}^{w_l}(u, v)}$ and $\hat{x}$ is assumed to be outside of $\mathrm{Disk}(u, v)$, $\mathrm{Disk}(u, v)$ intersects $\mathrm{C}(u, w_l, w_{l-1})$ two more times in $\overline{\mathrm{H}^{w_l}(u, v)}$, which contradicts elementary geometry. From this we conclude that $\hat{x}$ must be contained in $\mathrm{Disk}(u, v)$. This implies that $\angle u\hat{x}v \geq \pi/2$. Moreover, because $w_l \notin \mathrm{Disk}(u, v)$, it follows that $w_{l+1}, \ldots, w_k, w_{k+1} \notin \mathrm{Disk}(u, v)$, which especially means that $y \notin \mathrm{Disk}(u, v)$. Thus, $\angle uw_lv < \pi/2$ holds. Hence, y is not the angle maximizing GPDT-node w.r.t. (u, v). But this contradicts our initial assumption and we can conclude that this case cannot occur. ■

IV. Conclusion

To the best of our knowledge, we present in this work the first reactive algorithm for planar Euclidean spanner construction under the unit disk network model. We thus positively answer the previously raised question whether this is possible at all (see e.g., [8] and [9]). The algorithm constructs the PDT-neighborhood of the executing node with an optimal number of messages, as only those neighbors transmit a message that belong to the PDT-neighborhood. Our approach can be used, e.g., in greedy recovery routines of localized geographic routing protocols, enabling the selection of recovery paths whose lengths differ only by a constant factor from the actual shortest paths in the underlying network graph. In this setting, this is the first approach providing such a guarantee at all.

Nevertheless, at least in the greedy recovery application, our protocol competes with the Rotational Sweep algorithm (RS) [9]. The latter reactively computes a recovery path transmitting only three messages per hop, but without providing a constant guarantee on this path's stretch. Our approach computes the entire PDT-neighborhood per hop, but provides the aforementioned constant guarantee on the spanning ratio. Thus, a tradeoff between routing path length and message complexity is being introduced.

This immediately raises the question if both is possible: a constant guarantee on recovery path length and constant number of message transmissions per hop. The recovery path examples provided in [9, Fig. 5 and Fig. 22] suggest the assumption that RS recovery paths containing non-GG edges are paths of the PDT subgraph. However, this is currently unknown and requires further investigation.

Our contributions may also be of use in other related contexts, where planar spanners could lead to further efficiency improvements, such as in localized multicast routing [12].

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