A continuous, local strategy for constructing a short chain of mobile robots*

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Abstract. We are given an arbitrarily shaped chain of n robots with fixed end points in the plane. We assume that each robot can only see its two neighbors in the chain, which have to be within its viewing range. The goal is to move the robots to the straight line between the end points, while each robot has to base the decision where to move only on the relative positions of its neighbors. Such *local* strategies considered until now are based on discrete rounds, where a round consists of a movement of each robot. In this paper, we initiate the study of continuous local strategies: The robots may perpetually observe the relative positions of their neighbors, and may perpetually adjust their speed and direction in response to these observations. We assume a speed limit for the robots, that we normalize to one, which corresponds to the viewing range. Our contribution is a continuous, local strategy that needs time $\mathcal{O}(min\{n, (OPT + d)\log(n)\})$. Here d is the distance between the two stationary end points, and OPT is the time needed by an optimal global strategy. As our strategy has the property that the robot that reaches its destination as the latest always moves with maximum speed, the same bound as above also holds for the distance travelled.

1 Introduction

We envision a scenario where two stationary devices (stations) with a limited communication radius are placed in the plane. In order to provide communication between them, n mobile robots are deployed which form a chain capable of forwarding communication packets. Equally to the stations, the robots have only a limited communication range. Assuming that the robots are arranged as an arbitrarily shaped, possibly winding chain in the beginning, the goal is to design and analyze a strategy for the mobile robots that minimizes the length of the chain. Each robot has to plan and perform its movement

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solely based on the positions of its neighbors in the chain, which are within a constant distance in the beginning— no global view, communication or long term memory is provided.

Unlike most strategies considered for similar problems, we want to use a continuous time model. Therefore, we are not given a classical round model, but rather all robots can perpetually and at the same time measure and adjust their movement paths, leading to curves as trajectories for the robots. Although this model fits to real applications [1] and has also interesting and important theoretical aspects, surprisingly, to our knowledge, it has only once been considered theoretically for a formation problem [2]. The authors give an algorithm which gathers robots in one point in finite time, but they do not give any further runtime bounds. One reason for not using a continuous time model might be that completely different techniques of analysis have to be applied than for usual discrete models. We are optimistic that the techniques for analysis which we develop in this paper have the potential to be applied to other continuous formation problems, e.g. the gathering problem.

We study a natural strategy, which we call MOVE-ON-BISECTOR: A robot moves in the direction of the bisector between its two neighbors at all times with maximum speed 1 (which equals the viewing range), until it reaches the straight line between them, and thenceforward stays on this straight line. We analyze this strategy and prove that it is valid in terms of connectedness: If neighbors are originally in distance at most 1, they will remain in distance at most 1 when performing the MOVE-ON-BISECTOR-strategy. Then we show bounds on the runtime needed until all robots are on the straight line between the stations. Since the robots move with velocity 1 as long as they have not reached the line between their neighbors, the finishing time is also the maximum distance a robot can travel.

In order to measure the quality of our strategy, we consider the time needed by our local strategy compared to the time needed by an optimal global strategy. The optimal global time is clearly lower bounded by the maximum distance between the initial robot positions and the final straight line. We will call this maximum distance *height* throughout the paper.

The problem at hand has been investigated before, but only in classical discrete models, which we describe below in the section on related work.

Our contribution We initiate the study of a continuous time model for local robot formation problems by constructing a short communication chain. Our main result states that the MOVE-ON-BISECTOR-strategy needs at most time $\mathcal{O}(min\{n, (OPT + d)\log(n)\})$, where OPT is the time needed by an optimal global strategy and d the distance between the two stations. This result implies an asymptotically optimal worst case time bound of $\mathcal{O}(n)$. In addition it shows that our continuous, local strategy is $\mathcal{O}(\log(n))$-competitive compared to a global optimal strategy, if d is sufficiently small.

Organization of the paper We begin the next section by formally introducing the model and the MOVE-ON-BISECTOR-strategy. Section 3 is dedicated to the analysis of the strategy. The analysis is divided into three major parts: in Section 3.1 we show that the MOVE-ON-BISECTOR-strategy maintains a valid chain, where the robots stay in distance 1 to their neighbors. Then we analyze the MOVE-ON-BISECTOR-strategy on input instances in which at least some robots are far away from their final destination. Clearly, also a global algorithm needs long to optimize those chains. In Section 3.3 we proceed to the main result of the paper. We show that input instances that are solved fast by an optimal global algorithm are also handled fast by MOVE-ON-BISECTOR. We conclude with some open questions.

Related work The problem of locally building short communication chains has been considered before in discrete settings. In [3] an intuitive strategy has been presented: robots move synchronously to the old mid position of their neighbors. With this strategy, the robots converge to the line between the stations in $\mathcal{O}(n^2 \log n)$ and $\Omega(n^2)$ steps. This result has been improved in [4] with a more sophisticated strategy that lets the robots come close to the line in only a linear number of steps in the worst case. Note that an optimal strategy might be a lot faster depending on the input instance and no bounds for this case are given. There is also some experimental work from the engineering point of view [1,5]. In [6] a local algorithm for a more general problem is considered: robots are distributed in the plane and have to shorten a communication network between several base stations. They have to base their decision on where to move in the next round on the relative position of all robots currently within distance 1.

Related problems are the gathering problem and the convergence problem, where robots are not required to form a line, but to gather in or to converge to some not predefined point in the plane. The focus is usually to limit the capabilities of the robots as far as possible and show that the task can be achieved in finite time, e.g. [7,8,9,10,11,12]. However, there are only few examples with provable runtime bounds such as [13], where an $\mathcal{O}(n^2)$ bound for a global algorithm is given. There are also some results for gathering [14] and convergence [15] with local view, but only one giving a runtime [16].

A strategy for the gathering problem which is similar to the MOVE-ON-BISECTOR-strategy has been considered in [2]. The authors also use the continuous time model, interpreting it as the limit of a discrete time model with the length of the time steps converging to 0. However, in contrast to our work it is only proven that it gathers the robots in finite time. None of the work above compares the runtime results of a specific instance to those of an optimal global algorithm on the same instance.

2 Problem description

We consider a set of $n+2$ robots $v_0, v_1, \ldots, v_{n+1}$ in the two dimensional euclidean plane $\mathbb{R}^2$. The robots v_0 and v_{n+1} are stationary and will be referred to as *base stations* or simply *stations*, while we can control the movement of the remaining n robots $v_1, v_2, \ldots, v_n$. At the beginning, the robots form a chain, where each robot v_i is neighbor of the robots v_{i-1} and v_{i+1}. The chain may be arbitrarily winding in the beginning. The goal is to optimize the length of the communication chain in a distributed way. We are constrained in that the robots have a limited viewing range, which we set to 1. The communication chain is therefore *connected* if and only if, for each two neighbors in the chain, the distance between them is less than or equal to 1. We assume that the chain is connected at the beginning, and we say that a strategy for the robots is *valid* if it keeps the chain connected.

The robots use only information from the current point of time (they are oblivious), share no common sense of direction and communicate only by observing the positions of their two neighbors. However, we require that the robots are able to distinguish their neighbors from the remaining robots in the communication chain (it is not necessary to distinguish the two neighbors from each other).

The continuous time model In our continuous time model, time passes in a continuous way and is not modeled by discrete time steps. Thus, robots are able to continuously measure the positions of their neighbors and adjust their trajectory and speed accordingly (keeping the speed limit of 1). In the MOVE-ON-BISECTOR-strategy, the direction in which a robot moves can change continuously. In contrast, the speed can also change rapidly in a non-continuous way. Both direction and speed depend only on the positions of the neighboring robots. We assume that a robot can measure these positions *without delay*: the direction in which a robot moves depends only on the positions of its neighbors at the same time.

In order to measure the quality of our algorithm, we determine the time until all robots have reached their final position. Assuming a maximum velocity of 1, this time is equal to the distance which is travelled by a robot which always moves with velocity 1. We will see that when using the MOVE-ON-BISECTOR-strategy, there is at least one robot for which this is true.

In our model we assume accuracy of the actors and sensors of the robots in several aspects: measurements of the relative positions of the neighbors of a robot (angle formed by a robot and its neighbours, distances to neighbors) are exact; adjustment of speed and direction is accurate; this adjustment is not delayed compared to the measurement. In the final Section 4 we will discuss our strategy in the light of inaccurate sensors and actors.

The MOVE-ON-BISECTOR-*Strategy* In the MOVE-ON-BISECTOR strategy, robots that have not yet reached the straight line between their two neighbors will move along the bisector formed by the vectors pointing towards their two neighbors with maximum speed 1 (phase 1). Once they have reached this line, they adapt their velocity to stay on this (moving) line keeping the ratio between the distances to their neighbors constant (phase 2). Since the neighbors are also restricted to the maximum speed of 1, this is always possible: a robot will not have to move faster than with speed 1 to stay on this line and to keep the ratio. One special situation can occur: If two neighboring robots v_i and v_j are at the same position at the same time, both take the other neighbor of v_j or v_i respectively as their new neighbor. Then, both robots have the same neighbors and will stay together from now on. For the sake of clarity, we will ignore such situations in the analysis.

Since robots which have reached the second phase stay in this phase until the end, all robots have reached the final line between the two stations as soon as all robots are in the second phase. Thus, the last robot reaching the second phase always moves with maximum speed.

Notions & Notation Given a time $t \geq 0$, the position of robot v_i at this time is denoted by $v_i(t) \in \mathbb{R}^2$. If not stated otherwise, we will assume $v_0(0) = (0,0)$ and $v_{n+1}(0) = (d,0)$,

$d \in \mathbb{R}_{\geq 0}$ denoting the distance between the two base stations. The vector connecting two neighboring robots v_{i-1} and v_i will be denoted by $w_i(t) := v_i(t) - v_{i-1}(t)$ for $i = 1, 2, \ldots, n+1$. By $\alpha_i(t) \geq 0$ we denote the smaller of the two angles formed by the vectors $-w_i(t)$ and $w_{i+1}(t)$. We will furthermore denote the scalar product of two vectors a and b simply by $a \cdot b$ and the length of a vector a by $||a||$. A fixed placement of the robots (say their positions at a given time t) is called a *configuration*. Furthermore, we define two properties for a given configuration:

Definition 1 (height). *The* height $\mathfrak{h}(t)$ *of a configuration at time t is the maximum distance between a robot in time t and the straight line between the stations.*

Definition 2 (length). *The* length $\mathfrak{l}(t)$ *of a configuration at a time t is defined as the sum of the distances between neighboring robots:* $\mathfrak{l}(t) := \sum_{i=1}^{n+1} ||w_i(t)||$.

Clearly, the height $h := \mathfrak{h}(0)$ is a lower bound for the time needed by an optimal global algorithm, since every algorithm needs to cover this distance. The length of a configuration is also a natural quantity to measure its quality: a winding chain is relatively long compared to a straight line. Since the distance between two robots may be at most 1, it holds that $h \leq \frac{l}{2}$ and $l \leq n+1$. See Figure 1 and Figure 2 for an illustration of the notions defined in this section.

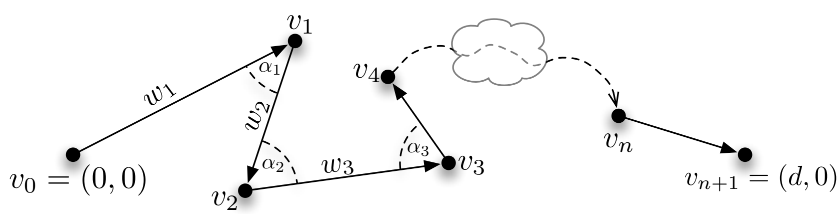


Fig. 1: Robots v_i positioned in the euclidean plane. For clarity we omitted the time parameter t.

3 Analysis

In this section, we will show that the MOVE-ON-BISECTOR-strategy is valid (Subsection 3.1) and analyze the time needed until all robots are positioned on the line between the two base stations (Subsections 3.2 and 3.3). We use the *length* and *height* of a given configuration to show two upper bounds of $\mathcal{O}(l)$ and $\mathcal{O}((h+d)\log l)$, where d denotes the distance between the two base stations. The first bound is tight for configurations, where an optimal global algorithm is slow. On other configurations this bound can be arbitrary bad. Therefore, we show the second bound for this kind of input instances and combine the bounds to our main result of an upper bound of $\mathcal{O}(min\{n, (OPT + d)\log(n)\})$.

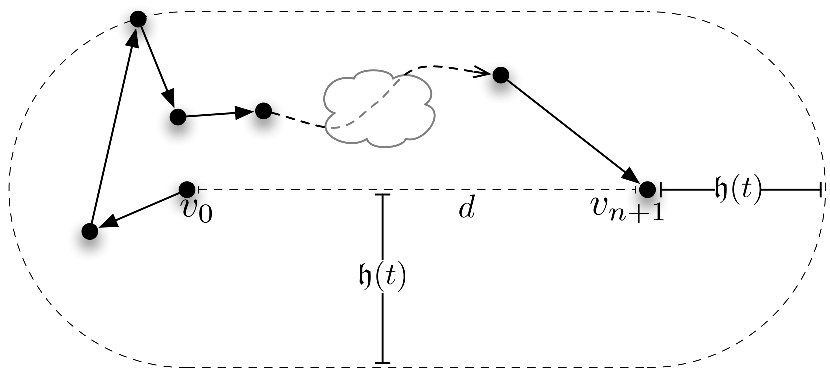


Fig. 2: Illustration of the height $\mathfrak{h}(t)$ of a configuration.

3.1 Validity of the strategy

Let us first consider two robots v_i and v_j with $j > i$ at a time when neither v_i nor v_j have reached the line between their neighbors, but any robot v_k with $i < k < j$ has. That is, the robots v_k form a straight line between v_i and v_j. We will show that the distance between v_i and v_j decreases with non-negative speed. Given that all robots v_k between v_i and v_j maintain the ratio between the distances to their corresponding neighbors, this implies that the distance between *any* two neighboring robots is monotonically decreasing, and thus the chain stays connected and the MOVE-ON-BISECTOR-strategy is valid. We start by considering the case that both, v_i and v_j, are *mobile* (not base stations).

Lemma 1. *Given two* robots v_i *and* v_j *at an arbitrary time* t_0*, their distance decreases with speed* $\cos\frac{\alpha_i(t_0)}{2} + \cos\frac{\alpha_j(t_0)}{2} \geq 0$.

Proof. We define $D : \mathbb{R}_{\geq 0} \to \mathbb{R}^2, t \mapsto v_j - v_i$ and $\mathsf{d} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, t \mapsto ||D(t)||$. That is, $D(t)$ is the vector from v_i to v_j and $\mathsf{d}(t)$ the distance between v_i and v_j at time t. We want to show that $\mathsf{d}'(t_0) = -\left(\cos\frac{\alpha_i(t_0)}{2} + \cos\frac{\alpha_j(t_0)}{2}\right)$ for an arbitrary but fixed point of time t_0. We will refer to the x- and y-component of $D(t) \in \mathbb{R}^2$ in the following by $D_x(t)$ and $D_y(t)$ respectively.

By translating and rotating the coordinate system, we can w.l.o.g. assume $v_i(t_0) = (0,0)$ and $v_j(t_0) = (\mathsf{d}(t_0), 0)$. Due to the definition of the MOVE-ON-BISECTOR strategy, the velocity vectors of v_i and v_j at time t_0 are given by:

$$v_i'(t_0) = \left(+\cos\frac{\alpha_i(t_0)}{2}, \pm\sin\frac{\alpha_i(t_0)}{2}\right)$$
$$v_j'(t_0) = \left(-\cos\frac{\alpha_j(t_0)}{2}, \pm\sin\frac{\alpha_j(t_0)}{2}\right)$$

See Figure 3 for an illustration.

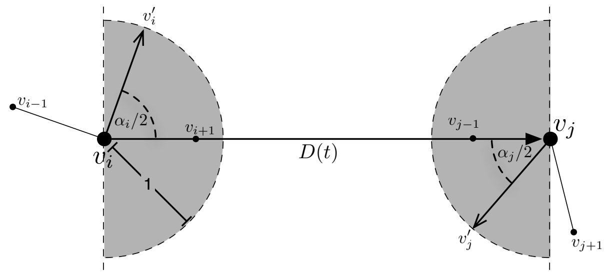


Fig. 3: Illustration of v_i's and v_j's velocity vectors v_i' and v_j'.

Basic analysis now gives us the following equation for the first derivation of d at a time $t \in \mathbb{R}_{\geq 0}$[1]:

$$\mathsf{d}'(t) = \begin{pmatrix} \frac{D_x(t)}{\mathsf{d}(t)} & \frac{D_y(t)}{\mathsf{d}(t)} \end{pmatrix} \cdot \begin{pmatrix} D_x'(t) \\ D_y'(t) \end{pmatrix}$$

Using that we have $D_y(t_0) = 0$ and $D_x(t_0) = \mathsf{d}(t_0)$ we finally get

$$\begin{aligned} \mathsf{d}'(t_0) = D_x'(t_0) = (v_j - v_i)'(t_0) = v_j'(t_0) - v_i'(t_0) \\ = \quad -\left(\cos\frac{\alpha_i(t_0)}{2} + \cos\frac{\alpha_j(t_0)}{2}\right) \end{aligned}$$

Therefore, the distance between v_i and v_j changes at time t with speed $\cos(\frac{\alpha_i(t_0)}{2}) + \cos(\frac{\alpha_j(t_0)}{2})$. Furthermore, since we have $\alpha_i(t) \in [0, \pi]$ for any $t \in \mathbb{R}_{\geq 0}$ and $i \in \{1, \ldots, n\}$, this speed is indeed positive and the distance *decreases*. □

A similar result holds if either v_i or v_j is a base station. Since this can be proven completely analogously to Lemma 1, we will omit the proof and merely state the corresponding result.

Lemma 2. *Consider two robots v_i and v_j at an arbitrary time t_0, one of them being a base station and the other a robot not yet having reached the line between its neighbors. Then their distance decreases with speed* $\cos\frac{\alpha_j(t_0)}{2} \geq 0$.

□

Now, we have the preliminaries to state the validity of the MOVE-ON-BISECTOR strategy.

Theorem 1. *The* MOVE-ON-BISECTOR *strategy is valid. That is, if the robot chain is connected at time t and all robots perform the* MOVE-ON-BISECTOR *strategy, the robot chain remains connected for any time $t' \geq t$.*

Proof. As described above, the statement follows immediately from Lemmas 1 and 2 together with the fact that any robot that has already reached the line between its neighbors will move such that it maintains the ratio between the distances to its two neighbors. □

[1] Remember that we assume $\mathsf{d}(t) \neq 0$ (see the description of the MOVE-ON-BISECTOR-strategy in Subsection 2).

3.2 The $\mathcal{O}(l)$ upper bound

We continue by analyzing how long it will take for all robots to reach the straight line between the two stations. We will derive a time bound of $\mathcal{O}(l)$, l denoting the length of the robots' initial configuration. Because $h \leq l/2$ and $l = \mathcal{O}(n)$ (the distance of neighboring robots is bounded by 1), this immediately implies a linear bound $\mathcal{O}(n)$ on the time until the optimal configuration is reached. Since there are start configurations with a height of $\Omega(n)$, the MOVE-ON-BISECTOR-strategy is asymptotically optimal for worst case start configurations. The next section will show a tighter bound for configurations, where the height is relatively small compared to the length of the configuration. The result can then be compared to an optimal algorithm.

In the following, we will show that either the length l or the height h of the robot chain decreases with constant speed. Since both are furthermore monotonically decreasing and bounded from below, this implies that the optimum configuration will be reached in time $\mathcal{O}(h+l) = \mathcal{O}(l)$. We begin with the monotonicity of the height.

Lemma 3. *The height of the robot chain is monotonically decreasing and bounded from below by 0.*

Proof. The lower bound is trivial, it follows directly from the definition of the robot chain's height. For the monotonicity, fix a time $t \in \mathbb{R}_{\geq 0}$ and consider the height $\mathfrak{h}(t)$ of the configuration at time t. Let B denote the line segment connecting both base stations and note that all robots are contained in the convex set $H := \{x \in \mathbb{R}^2 \mid dist(x,B) \leq \mathfrak{h}(t)\}$ of points having a distance of at most $\mathfrak{h}(t)$ to B. Let us consider an arbitrary robot v_k and its neighbors v_{k-1} and v_{k+1}. Since H is convex and all three robots lie in H, so does the bisector along which v_k moves. That is, v_k can not leave H. Since this argument applies to any robot, none of the robots can increase their distance to B beyond $\mathfrak{h}(t)$. This implies the monotonicity of the robot chain's height. $\square$

Lemma 4. *The length of the robot chain decreases with speed* $2\sum_{i=1}^{n} \cos\frac{\alpha_i(t)}{2}$ *and is bounded from below by* d.

Proof. Since both base stations do not move, the length can obviously not fall below their distance d. Using the function $\mathfrak{l} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, t \mapsto \mathfrak{l}(t)$ to refer to the chain's length at time t, it remains to show that $\mathfrak{l}'(t) = -2\sum_{i=1}^{n} \cos\frac{\alpha_i(t)}{2}$.

Fix a time $t \in \mathbb{R}_{\geq 0}$ and consider the robots $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ (for an $r \in \mathbb{N}$ and $i_s < i_{s+1} \forall s = 1, \ldots, r-1$) that have not yet reached the line between their neighbors. We make two observations:

- For any robot v_j of the remaining robots, it holds that $\alpha_j(t) = \pi$ and therefore $\cos\frac{\alpha_j(t)}{2} = 0$.
- Any of the remaining robots either lies on the line between some v_{i_s} and $v_{i_{s+1}}$ or on the line between one of the base stations and v_{i_1} or v_{i_r}. That is, setting $\mathfrak{l}_0(t) := ||v_0(t) - v_{i_1}(t)||$, $\mathfrak{l}_k(t) := ||v_{i_k}(t) - v_{i_{k+1}}(t)||$ $(k = 1, \ldots, r-1)$ and $\mathfrak{l}_r(t) := ||v_{i_r}(t) - v_{n+1}(t)||$, the length $\mathfrak{l}(t)$ of the chain is given by:

$$\mathfrak{l}(t) = \sum_{k=0}^{r} \mathfrak{l}_k(t)$$

Now, Lemmas 1 and 2 give us the derivations of these $\mathfrak{l}_k$, and therefore we have:

$$\mathfrak{l}'(t) = \mathfrak{l}'_0(t) + \sum_{k=1}^{r-1} \mathfrak{l}'_k(t) + \mathfrak{l}'_r(t)$$

$$= -\cos\frac{\alpha_{i_1}(t)}{2} + \sum_{k=1}^{r-1}(-\cos\frac{\alpha_{i_k}(t)}{2} - \cos\frac{\alpha_{i_{k+1}}(t)}{2}) - \cos\frac{\alpha_{i_r}(t)}{2}$$

$$= -2\sum_{k=1}^{r}\cos\frac{\alpha_{i_k}(t)}{2} = -2\sum_{i=1}^{n}\cos\frac{\alpha_i(t)}{2}.$$

□

Now we can prove an upper bound for the travelled distance in dependency of h and l, implying also a worst case upper bound.

Theorem 2. *When the* MOVE-ON-BISECTOR *strategy in the continuous model is performed, the maximum distance travelled by a robot is $\frac{\sqrt{2}}{2}h + \sqrt{2}l$, where h is the height and l the length of the robot chain in the start configuration.*

Proof. We will prove that in time $\frac{\sqrt{2}}{2}h + \sqrt{2}l$ all robots have reached their corresponding end positions. Given that the robots move with a maximum velocity of 1, this proves the theorem. To do so, we show that at any time, either the height function $\mathfrak{h} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ or the length function $\mathfrak{l} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are strictly decreasing *by a constant factor*. Together with Lemmas 3 and 4 (the monotonicity and non-negativity of $\mathfrak{l}$ and $\mathfrak{h}$) this proves the theorem.

So, let us consider an arbitrary time $t \in \mathbb{R}_{\geq 0}$. We distinguish two cases:

Case 1: $\exists i \in \{1,\dots,n\} : \alpha_i(t) \leq \pi/2$

In this case, Lemma 4 states that:

$$\mathfrak{l}'(t) = -2\sum_{k=1}^{n}\cos\frac{\alpha_k(t)}{2} \leq -2\cos\frac{\alpha_i(t)}{2} \leq -2\cos\frac{\pi}{4} = -\sqrt{2}$$

That is, the length of the robot chain decreases with a *constant* speed of at least $\sqrt{2}$.

Case 2: $\forall i \in \{1,\dots,n\} : \alpha_i(t) > \pi/2$

Using the terms from the proof of Lemma 3, consider a robot v_k with distance $\mathfrak{h}(t)$ to the line segment B connecting both base stations. Align the coordinate system such that the line L through $v_k(t)$ having distance $\mathfrak{h}(t)$ to B corresponds to the x axis and $v_k(t)$ to the origin. Figure 4 illustrates the situation.

We know that both neighbors of v_i must lie on the same side of L as B, w.l.o.g. let it be the lower side. Furthermore, because we have $\alpha_k(t) > \pi/2$, one neighbor must lie to the lower left and the other to the lower right of v_k. This implies that v_k's velocity vector is directed downwards, forming an angle of less than $\pi/4$ with the y-axis. Therefore, v_k moves with a speed of more than $\cos\frac{\pi}{4}$ downwards.

Since this holds for any extremal robot, we get $h'(t) < -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$. That is, the height of the robot chain decreases with a constant speed of at least $\frac{\sqrt{2}}{2}$. □

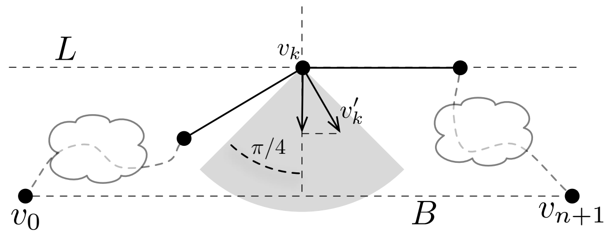


Fig. 4: If all angles α_i are larger than $\pi/2$, then the velocity vector of a "highest" robot v_k lies within the gray area. It therefore moves downwards with a speed of at least $\cos\pi/4$.

Since $h \in \mathcal{O}(l)$, Theorem 2 gives an upper bound of $\mathcal{O}(l)$ for arbitrary start configurations. This result directly shows that the MOVE-ON-BISECTOR-strategy is asymptotically optimal for worst-case instances (Cor. 1), the measure which is usually used in the literature. Still, this bound can be arbitrarily worse than an optimal algorithm on specific instances. We will investigate these instances in the next section.

Corollary 1. *When the* MOVE-ON-BISECTOR-*strategy in the continuous model is performed, the maximum distance travelled by a robot is $\Theta(n)$ for a worst-case start configuration.*

Proof. Obviously it holds that $h \leq \frac{l}{2} \leq \frac{n+1}{2}$. For the lower bound, we can use a start configuration in which the stations share the position $(0,0)$ and $v_i(0) = v_{n+1-i}(0) = (0,i)$. Thus, the robot in the middle of the chain is in distance $\approx \frac{n}{2}$ of its end position and MOVE-ON-BISECTOR (as well as any global algorithm) needs at least this time until all robots have reached the line between the base stations. □

3.3 The $\mathcal{O}((h+d)\log l)$ upper bound

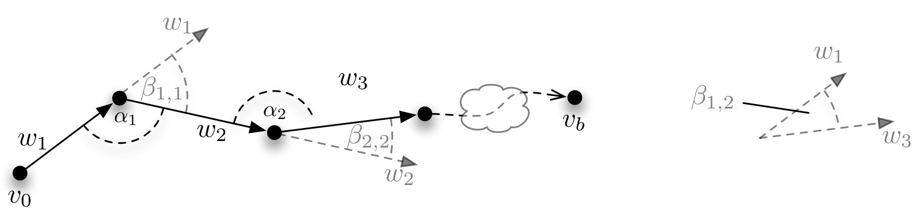
Assume we are given a configuration whose height is—relative to the length of the communication chain—very small. In this case, the upper bound of $\mathcal{O}(l)$ for our strategy can be arbitrarily larger than the time needed by an optimal strategy, which can be as small as h. But intuitively, given a long chain with a small height, the chain must be quite winding, yielding many relatively small angles α_i. The result is that the chain length does not only decrease at one robot, as we can only guarantee for arbitrary configurations, but there are many robots which reduce the length of the chain (Lemma 4).

For the proof of this upper bound, we will divide the chain into parts of length $\Theta(h+d)$ and show that each part must contain some curves. In particular, in each part, the sum of the angles $\alpha_i(t)$ must be by a constant smaller than in a straight line (Lemma 5). Lemma 6 transfers this result for each part to the sum of the angles of the whole chain. Having that the sum of the angles in the whole chain cannot be arbitrarily large, Lemma 7 yields the speed by which the length of the chain decreases. Since the number of parts is dependent on the length of the chain, the speed is also dependent on it. Theorem 3 finally gives the upper bound of $\mathcal{O}((d+h)\log l)$.

Lemma 5. *Let $\mathfrak{B}$ denote an arbitrary rectangular box containing the robots $v_{a-1}, v_a, v_{a+1}, \ldots, v_b$ (for $a, b \in \{1, \ldots, n+1\}$, $a < b$) at a given time $t \in \mathbb{R}_{>0}$ and let S be the diagonal length of the box. Then we have:*

$$\sum_{k=a}^{b} ||w_k(t)|| \geq \sqrt{2} \cdot S \Rightarrow \sum_{k=a}^{b-1} \alpha_k(t) \leq (b-a)\pi - \frac{\pi}{3}$$

Proof. For the sake of clarity, we will omit the time parameter t in the following. That is we write α_k, v_k and w_k instead of $\alpha_k(t)$, $v_k(t)$ and $w_k(t)$. Furthermore, we assume w.l.o.g. $a = 1$. Thus, we have to show $\sum_{k=1}^{b} ||w_k|| \geq \sqrt{2} \cdot S \Rightarrow \sum_{k=1}^{b-1} \alpha_k \leq (b-1)\pi - \frac{\pi}{3}$



(a) (b)

Fig. 5: Note that the angles $\beta_{i,j}$ are signed, e.g.: $\beta_{1,1} > 0$, $\beta_{2,2} < 0$, $\beta_{1,2} = \beta_{1,2} + \beta_{2,3} > 0$.

Consider the function $\angle : \mathbb{R}^2 \times \mathbb{R}^2 \to]-\pi, \pi]$ that maps two vectors (w_i, w_j) to the signed angle of absolute value $\leq \pi$ formed by them (it is not important which direction is used as positive angle, as long as it is equal for all pairs of vectors (w_i, w_j)). Note that we have $\alpha_k = \pi - |\angle(w_k, w_{k+1})|$ for all $k = 1, \ldots, b-1$. Let us define $\beta_{i,j} := \sum_{k=i}^{j} \angle(w_k, w_{k+1})$ and observe that $\angle(w_i, w_j) \equiv \beta_{i,j} \mod]-\pi, \pi]$. See Figure 5 for an illustration.

Let us now assume $\sum_{k=1}^{b} ||w_k(t)|| \geq \sqrt{2} \cdot S$ and consider the following two cases:

Case 1: $\exists i, j, 1 \leq i < j \leq b : |\beta_{i,j}| \geq \frac{\pi}{3}$

Intuitively, if the angle between two vectors in the chain is large, the sum of the inner angles α_k of the robots in between cannot be arbitrarily large. More formally,

$$\begin{aligned}
\sum_{k=1}^{b-1} \alpha_k &\leq (i-1)\pi + \sum_{k=i}^{j} \alpha_k + (b-1-j)\pi \\
&= (b+i-j-2)\pi + \sum_{k=i}^{j} (\pi - |\angle(w_k, w_{k+1})|) = (b-1)\pi - \sum_{k=i}^{j} |\angle(w_k, w_{k+1})| \\
&\leq (b-1)\pi - \left| \sum_{k=i}^{j} \angle(w_k, w_{k+1}) \right| = (b-1)\pi - |\beta_{i,j}| \leq (b-1)\pi - \frac{\pi}{3}
\end{aligned}$$

Thus, the lemma holds in this case.

Case 2: $\forall i,j, 1 \le i < j \le b : |\beta_{i,j}| < \frac{\pi}{3}$

We will show that this case cannot occur by showing that the vector connecting v_0 and v_b, which is equal to $\sum_{k=1}^{b} w_k$, would have to be longer than S, which is a contradiction to v_0 and v_b both lying in $\mathfrak{B}$.

We have $\angle(w_i, w_j) = \beta_{i,j}$ and $|\beta_{i,j}| < \frac{\pi}{3}$ for all $1 \le i < j \le b$. In the following, we will use that the squared length of a vector is equal to its scalar product with itself. Therefore:

$$\begin{aligned}
\left|\left|\sum_{k=1}^{b} w_k\right|\right|^2 &= \left(\sum_{k=1}^{b} w_k\right) \cdot \left(\sum_{k=1}^{b} w_k\right) = \sum_{1\le i,j\le b} w_i \cdot w_j = \sum_{1\le i,j\le b} ||w_i|| \cdot ||w_j|| \cdot \cos(\beta_{i,j}) \\
&> \sum_{1\le i,j\le b} ||w_i|| \cdot ||w_j|| \cdot \cos(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) \sum_{1\le i,j\le b} ||w_i|| \cdot ||w_j|| \\
&= \frac{1}{2}\left(\sum_{k=1}^{b} ||w_k||\right)^2 \ge \frac{1}{2} \cdot (\sqrt{2}S)^2 = S^2
\end{aligned}$$

This implies $||\sum_{k=1}^{b} w_k|| > S$, leading to the desired contradiction.

$\square$

Dividing the chain in parts of length at least $\sqrt{2}$ times the diagonal of the height box, Lemma 5 shows that each of the parts must contain some ”small” angles. The robots at these angles therefore shorten the length of the chain. The following lemma shows that using the technique of dividing the chain into parts yields an upper bound on the sum of the angles α_i of the chain.

Lemma 6. *Let S denote the diagonal length of the robots' height-box at a given time t. Then we have:*

$$\sum_{k=1}^{n} \alpha_k(t) \le n\pi - \frac{\pi}{3}\left\lfloor \frac{\mathfrak{l}(t)}{2\sqrt{2}S} \right\rfloor$$

Proof. As in the proof for Lemma 5, we will omit the time parameter t in the following.

First note that we have $||w_k|| \le S$, because all robots lie inside the height-box. This allows us to recursively define indices $1 = a_0 < a_1 < \ldots < a_m \le n+1$ by demanding $a_i \in \mathbb{N}$ to be minimal with $\sum_{k=a_{i-1}}^{a_i} ||w_k|| \in [\sqrt{2}S, (\sqrt{2}+1)S[$. That is, we divide the chain at time t in m parts, where $v_{a_{i-1}}$ and v_{a_i} bound part i. v_{a_i} is the first robot in the chain such that the length of part i is at least $\sqrt{2}S$. Furthermore, since $||w_{a_i}|| \le S$, the length of part i is at most $\sqrt{2}S + S \le 2\sqrt{2}S$, which implies $m \ge \left\lfloor \frac{\mathfrak{l}}{2\sqrt{2}S} \right\rfloor$. Since we have $\sum_{k=a_{i-1}}^{a_i} ||w_k|| \ge \sqrt{2}S$ for all $i = 1, \ldots, m$, by Lemma 5 we get $\sum_{k=a_{i-1}}^{a_i-1} \alpha_k \le (a_i - a_{i-1})\pi - \frac{\pi}{3}$. We now compute:

$$\sum_{k=1}^{a_m-1}\alpha_k = \sum_{i=1}^{m}\sum_{k=a_{i-1}}^{a_i-1}\alpha_k \le \sum_{i=1}^{m}\left((a_i-a_{i-1})\pi-\frac{\pi}{3}\right)$$
$$=\pi\sum_{i=1}^{m}(a_i-a_{i-1})-\frac{\pi}{3}m=(a_m-a_0)\pi-\frac{\pi}{3}m$$
$$=(a_m-1)\pi-\frac{\pi}{3}m$$

This implies $\sum_{k=1}^{n}\alpha_k \le n\pi-\frac{\pi}{3}m \le n\pi-\frac{\pi}{3}\left\lfloor\frac{\mathfrak{l}}{2\sqrt{2}S}\right\rfloor$, as the lemma states. □

Using that the sum of the angles α_i is bounded, we can now give a lower bound for the speed by which the chain length decreases, which is linear in the current number of parts and therefore the length of the chain. Instead of the current number of parts, which cannot be determined exactly only knowing the length of the chain, we use a lower bound for the number of parts.

Lemma 7. *The length of the robot chain decreases at least with speed* $\frac{2}{3}\left\lfloor\frac{\mathfrak{l}(t)}{2\sqrt{2}S}\right\rfloor$.

Proof. Fix a time $t\in\mathbb{R}_{\ge 0}$. By Lemma 4, the chain length decreases with a speed of $2\sum_{k=1}^{n}\cos\frac{\alpha_k(t)}{2}$. Using that $\cos(x)$ is lower bounded by $1-\frac{2}{\pi}x$ for all $x\in[0,\pi/2]$ and by Lemma 6 we get:

$$\mathfrak{l}'(t)=-2\sum_{k=1}^{n}\cos\frac{\alpha_k(t)}{2}\le -2\sum_{k=1}^{n}\left(1-\frac{\alpha_k(t)}{\pi}\right)=-2n+\frac{2}{\pi}\sum_{k=1}^{n}\alpha_k(t)$$
$$\le -2n+\frac{2}{\pi}\left(n\pi-\frac{\pi}{3}\left\lfloor\frac{\mathfrak{l}(t)}{2\sqrt{2}S}\right\rfloor\right)=-\frac{2}{3}\left\lfloor\frac{\mathfrak{l}(t)}{2\sqrt{2}S}\right\rfloor.$$

□

Now we can finally state our main result.

Theorem 3. *When the* MOVE-ON-BISECTOR *strategy in the continuous model is performed, the maximum distance travelled by a robot is* $\mathcal{O}((h+d)\log(l))$*, where* h *is the height and* l *the length of the robot chain in the start configuration.*

Proof. Set $m^* := \left\lfloor\frac{l}{2\sqrt{2}S}\right\rfloor$ and let us define m^* time-phases $\mathfrak{p}_i := [t_{i-1},t_i]$ for $i=1\dots,m^*$ by setting $t_0:=0$ and t_i for $i>0$ to the time when we have $\mathfrak{l}(t_i)=(m^*-i+1)\cdot 2\sqrt{2}S$. That is, during one phase $\mathfrak{p}_i$, the chain length is reduced by exactly $2\sqrt{2}S$ and thus in phase i, the chain must consist of at least m^* parts as defined in Lemma 6. Note that these t_i are well-defined, because by Lemma 7, in phase $\mathfrak{p}_i$ the chain length decreases with a speed of at least $\frac{2}{3}\left\lfloor\frac{\mathfrak{l}(t_i)}{2\sqrt{2}S}\right\rfloor=\frac{2}{3}\cdot(m^*-i+1)$ (which is a constant for fixed i). Furthermore, Lemma 7 gives us an upper bound on the length of each single phase $\mathfrak{p}_i$:

$$t_i-t_{i-1}\le\frac{\mathfrak{l}(t_{i-1})-\mathfrak{l}(t_i)}{\frac{2}{3}(m^*-i+1)}\le\frac{2\sqrt{2}S}{\frac{2}{3}(m^*-i+1)}$$

This allows us to give an upper bound to the time when the last phase ends:

$$\begin{aligned} t_{m^*} &= \sum_{i=1}^{m^*}(t_i - t_{i-1}) \leq 3\sqrt{2}S\sum_{i=1}^{m^*}\frac{1}{m^*-i+1} \\ &= 3\sqrt{2}S\sum_{i=1}^{m^*} i^{-1} < 3\sqrt{2}S \cdot (\ln m^* + 1) \end{aligned}$$

Now consider the situation after time $t \geq t_{m^*}$. We have $\mathfrak{l}(t_{m^*}) = (m^* - m^* + 1)2\sqrt{2}S = 2\sqrt{2}S$. By Theorem 2, from now on it takes time at most $\frac{\sqrt{2}}{2}\mathfrak{h}(t_{m^*}) + \sqrt{2}\mathfrak{l}(t_{m^*}) \leq \frac{\sqrt{2}}{2}h + 4S$ for the robots to reach the optimal configuration. Together with the bound on t_{m^*} and with $S = \mathcal{O}(h+d)$, this yields a maximum time (and therefore travelled distance) of

$$\begin{aligned} &3\sqrt{2}\cdot S\cdot(\ln m^* + 1) + \frac{\sqrt{2}}{2}h + 4S \\ \leq &3\sqrt{2}\cdot S\cdot\left(\ln\left(\frac{l}{2\sqrt{2}S}\right) + 1\right) + \frac{\sqrt{2}}{2}h + 4S \\ = &3\sqrt{2}\cdot S\cdot(\ln l - \ln(2\sqrt{2}S) + 1) + \frac{\sqrt{2}}{2}h + 4S \\ = &\mathcal{O}(S\cdot\ln l) + \frac{\sqrt{2}}{2}h + 4S = \mathcal{O}((h+d)\ln l) \end{aligned}$$

until the optimal configuration is reached. □

Corollary 2. MOVE-ON-BISECTOR *runs in time* $\mathcal{O}(min\{n, (OPT+d)\log n\})$.

□

A consequence of this result is that for $d \in \mathcal{O}(h)$ our local algorithm is by at most a logarithmic factor slower than an optimal global algorithm.

4 Outlook

We initiated the study of the continuous time model for the robot chain problem. Furthermore we introduced the idea to compare a local algorithm to an optimal global algorithm for the same instance.

We showed the runtime of our algorithm to be $\mathcal{O}(min\{n, (OPT+d)\log(n)\})$. Future work includes improving our upper bounds for the algorithm as well as lower bounds for optimal global algorithms. Furthermore, we want to apply the developed techniques for the continuous time model and the concept of comparing local algorithms to optimal global ones to further formation problems, such as gathering and convergence to a point in the plane as well as more complex tasks like building short two-dimensional communication infrastructures like trees.

It is an interesting problem to investigate to which extend our strategy is robust under inaccuracies of sensors and actors as described in the chapter on the continuous time model in Section 1. How to formally model such inaccuracies? For which input instances is our strategy robust? Do we have to modify the strategy? We certainly have to assume that input instances have the property that neighboring robots have distance at most $1-\gamma$, where $\gamma \in (0,1)$ is chosen dependent on parameters describing the accuracy.

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