Local Strategies for Maintaining a Chain of Relay Stations between an Explorer and a Base Station *

(extended abstract)†

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ABSTRACT

We discuss strategies for maintaining connectivity in a system consisting of a stationary base station and a mobile explorer. For this purpose we introduce the concept of mobile relay stations, which form a chain between the base station and the explorer and forward all communication.

In order to cope with the mobility of the explorer, relay stations must adapt their positions. We investigate strategies which allow the relay stations to self-organize in order to maintain a chain of small length. For a plane without obstacles, the optimal positions are on a line connecting the base station with the explorer; in a setting with obstacles it is a curve around some of the obstacles. Our goal is to keep the relay stations as close to this line/curve as possible. A crucial requirement for strategies is that they are able to work with imprecise or without localization and odometry information. Furthermore, strategies should be local, i.e., relay stations should not need to know about the state of the system as a whole. The performance measures for strategies are the number of relay stations used (in comparision to the optimal number) and the allowed speed of the explorer (in comparision to its maximum attainable speed).

We contribute by analyzing the performance of an already known strategy *Go-To-The-Middle*. This strategy assumes a very weak robot model and needs hardly any localization information, but sacrifices perfomance. Our main contribution is a new strategy, the *Chase-Explorer* strategy, and its analysis. It needs more advanced robots than *Go-To-The-Middle*, but achieves near-optimal performance. We further extend it to exploring terrains with obstacles.

Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Geometrical problems and computations; C.2.4 [**Distributed Systems**]

General Terms

Algorithms, Performance, Theory

Keywords

swarm robotics, self-organization, distributed algorithms, ad-hoc networks

1. INTRODUCTION

We envision a robotic scenario with a mobile explorer and a stationary base station. The explorer moves on a terrain with or without obstacles, guided by its own algorithm. The base station requires a stable communication link to the explorer in order to periodically communicate high-level goals and check the status of the explorer. In a large terrain with natural obstacles (e.g. mountains) it is hard to maintain a direct connection between the explorer and the base station with wireless radio devices. In this paper we are dealing with the question on how to guarantee such a communication link. Although satellite connections can be a solution

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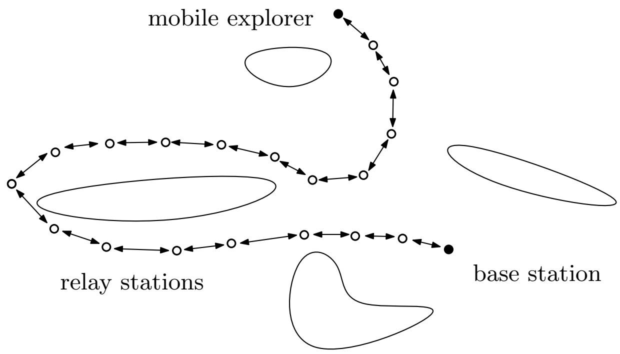

Figure 1: A non-optimal "loose" chain

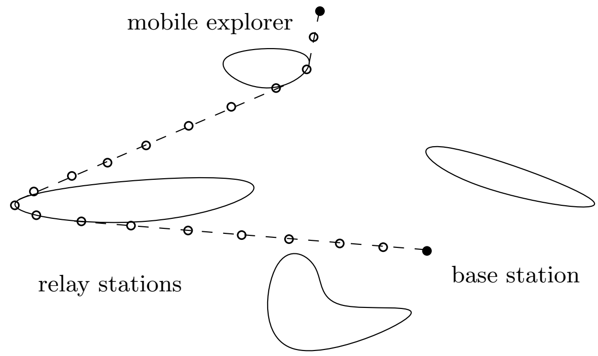

Figure 2: Topologically restricted optimal chain

to this problem, they are not available to everyone, costly and impose large energy requirements on radio transceivers. Another possibility of coping with this problem is to use mobile relay stations, which act as repeaters for the wireless signal. These relay stations are placed in the terrain in critical places, as shown on Fig. 1. Since the position of the explorer changes continously, it is a non-trival task to to organize the relay stations so that they are able to forward the radio signals properly.

The approach of using mobile relay stations is quite novel. First work on foundations in this area has been recently proposed by us in [8]. Some engineering work on a very similar scenario has been shown in [9]. The robot model we assume is similar to those presented in literature on coordination in robot swarms. This model imposes certain limitations on the strategies we may use for organizing the communication chain: the strategies should be local, i.e., a relay station should care only about its local environment and we assume that localization methods (either local between stations or global) are extremely unprecise.

We assume a worst-case adversarial scenario for the movement pattern of the explorer. This implies that we do not have any knowledge about the explorer's future movement except its maximum movement speed. This is similar to the approach presented in [11]. Thus, the presented results do not rely on any models of behavior in mobile networks. This allows the proposed system to be applied for any activity in a terrain like exploration or rescue actions.

There are three main goals when maintaining the communication chain. First, stations neighbored in a chain should always stay in transmission distance (fixed to $\mathcal{R}$ here). Second, the number of relay stations used should be as small as possible, since they are expected to be a rare resource. At last, the movement of the explorer should not be hindered by the chain – this might happen if the chain reacts too slowly to the movement of the explorer, and the explorer is forced to slow down in order not to exceed the distance $\mathcal{R}$ to the first relay station.

In a terrain without obstacles, the communication chain between the base station and the explorer should optimally be arranged on a straight line. On the other hand, since the explorer moves steadily, it is impossible to maintain a really straight line all the time without using fast multi-hop communication and knowing the exact position of the explorer relatively to the base station.

A terrain with obstacles is expected to be a plane with obstacles represented by points. The situation here is more complicated. There always exists (at least one) shortest path between the current position of the explorer and the base station. It is though unrealistic that the shortest path could be maintained by any local strategy – due to obstacles, a small movement of the explorer may completely change the topology of the optimal path. Thus, the goal for our strategy is to arrange the relay stations on a path which is the shortest path between the explorer and the base station topologically equivalent to the movement path of the explorer. Fig. 2 shows an example of such an optimal, topologically restricted, path.

1.1 Problem Model

We model the terrain as a plane with the Euclidean distance measure. The base station, relay stations, the explorer and obstacles are modeled as points on the plane. The position of both the explorer and the relay stations may not fall into the area of any of the obstacle.

The relay stations execute their strategy in *Look-Compute-Move* steps (therefore the model is named the *LCM*-model). In the first operation of the step the robot gathers new information about its environment by scanning it with its sensors. In the second operation, the sensoric input is analyzed and a decision is made about the behavior of the robot within the current step. During the last operation, the robot moves to a precomputed position. Note that, in contrast to our intuition, the *Move* operation is not necessarily that one which takes the longest time. Very often, gathering the sensoric input is even more time-consuming, if for example laser scans of the surrounding must be taken. For some scenarios we will assume a slightly stronger model, than the mentioned LCM. We then allow the robots to introduce a *Communication* operation at the end of each step. A crucial property of this model is that the robots must prepare the message to be sent during the *Compute* operation and so can react on a message received only one step after receiving it.

Different types of synchronization models are typically used to describe the abilities of robots acting in a swarm. We adopt the notions by [6]. Our work is based on the $\mathcal{FSYNC}$ model, in which all *LCM*-steps are fully synchronized. That means, that if a robot finishes moving earlier then others, it waits until the rest is finished (this may be made explicit by inserting a *Wait* operation into the definition of a step).

We bound the speed of our robots. The explorer is able to move by at most one distance unit per time step. Each of the relay stations is able to move by at least one distance unit per time step if such a movement is requested by its strategy. This can be seen as a lower bound on the movement abilities of the relay stations.

The transmission distance of the wireless transceivers is given by a constant $\mathcal{R} > 2$. We say that two stations can communicate when the distance between them is at most $\mathcal{R}$. This is the typical communication model known as a (unit) disc graph.

We arrange relay stations in a chain which spans from the explorer to the base station. Relay station are denoted $v_1, \ldots, v_{n_t}$, starting with v_1 at the relay station next to the explorer. The last relay station in time step t (the number of stations may vary depending on the time step) is denoted by v_{n_t}. We will often use v_0 to denote the explorer and v_{n_t+1} to denote the base station. For relay station v_i its neighbor in the direction of the explorer is v_{i-1} and the other one is v_{i+1}. Let $p_t(i)$ be the position of station v_i at the beginning of time step t. We set $b := p_t(n_t + 1)$ to be the position of the base station.

A strategy solving the relaying problem must ensure that a valid multi-hop communication path between the base station and the explorer is maintained all over the time, so that neighbored stations in a chain must be within distance $\mathcal{R}$. More formally, $|p_t(i) - p_t(i-1)|_2 \leq \mathcal{R}$ for all $1 \leq i \leq n_t + 1$.

1.2 Robot model

In this paper we investigate two types of robots. For the *Go-To-The-Middle* strategy the robots are simpler and have the following abilities

- ability to sense the relative position of other robots in distance at most $\mathcal{R}$,
- odometry and local measurements are precise,
- no memory, the robots are oblivious,
- no communication between robots is allowed,
- all robots execute the same strategy and have no IDs.

For the *Chase-Explorer* strategy the robot type has the following abilities

- ability to sense the relative position of other robots in distance at most $\mathcal{R}$,
- odometry and local measurements are imprecise, with a multiplicative error of ϵ,
- the robots posses memory,
- the robots are allowed to communicate with each other, but information received from a neighbor in step t may be only propagated to the other neighbor in step $t+1$,
- all robots execute the same strategy.

1.3 Performance Measures

We consider two measures for the performance of a strategy maintaining the communication chain. The first measure is the number of stations used, comparing to the number of relay stations necessary in an optimal chain.

The second measure is a guaranteed speed lower bound for the explorer. In certain types of strategies the explorer has to introduce wait cycles or to slow down, in order to allow for the communication chain to catch up. This introduces a lower bound on the average speed the explorer can attain and thus decreases the performance of the system as a whole.

1.4 Related Work

The amount of literature on communication in mobile wireless networks has developed significantly during last years. Most of this work considers connected networks in which the primary goal is to organize communication in the network despite the mobility.

The problem we are studying is similar to the efforts to ensure connectivity in a mobile network with the use of mobile backbones (e.g. [14]) and topology control (e.g. [10, 15, 1]). On the other hand, the concept of backbones provides only solutions for dense, connected networks, where the main goal is to select an easily manageable subset of edges/vertices from the connection graph. We are dealing with a mobile communication structure which is designed to work in larger, sparse networks, where it is necessary to control the mobility of the chain to ensure connectivity.

Similar problems related to robots with low capabilities have been considered in work on swarm robotics. Strategies have been developed to let robots gather [5, 6, 7, 2, 4] and on forming various geometrical patterns [3, 12, 13]. Note that most of this work assumes that robots are allowed to form patterns anywhere on the terrain, whereas we want them to form a line (or curve) in a specific place.

In [8] we have first presented the *Go-To-The-Middle* strategy. We have analyzed it in a setting with a stationary explorer, connected to the base station by a relay station chain forming an arbitrary curve. We have shown that starting with such an arbitrary configuration, the relay stations reach positions near to optimal after $\mathcal{O}(n^2 \log n)$ steps, where n is the number of relay stations in the start configuration. This result is obtained by reducing the problem to a stochastic process (similar to a Markov chain) and bounding its convergence speed.

1.5 Our Contribution

In Section 2 we provide new insights into the *Go-To-The-Middle* strategy (introduced in [8]). In particular, we provide a lower bound on its performance in a dynamic setting (where the explorer moves), showing that there are scenarios where the speed of the explorer drops to $\mathcal{O}(1/d)$ where d is the distance of the explorer to the base station (see Theorem 1). This shows that the most intuitive strategy for the proposed problem has serious performance drawbacks, as the speed of the explorer drops with its distance to the base station. We also show an argument which indicates that this lower bound on the performance is tight, i.e. that the *Go-To-The-Middle* strategy is able to achieve that explorer speed in certain circumstances.

In Section 3 we define the *Chase-Explorer* strategy and analyze its performance. This strategy does not restrict the movement of the explorer at all, i.e, the explorer is able to move with its maximum speed all the time, without having to care whether the chain of relay stations keeps up (see Theorem 6). Furthermore, the strategy uses only 1.5 times the optimal number of relay stations, if the strategy is provided with a sufficiently exact localization (see Theorem 7).

We also provide a localization scheme which works with our relay station chains and provides sufficiently precise measurements for our strategies to work properly. It depends only on local, imprecise measurements between neighbored relay stations. It furthermore allows the explorer to obtain a rough estimate of of its relative position to the base station, without using any localization infrastructure (like GPS) and odometry.

Section 4 extends the *Chase-Explorer* strategy for a scenario with obstacles. For this scenario, we show that the the speed of the explorer is inversely proportional to the complexity of the terrain, i.e., the number of obstacles.

2. GO-TO-THE-MIDDLE STRATEGY

We briefly describe the *Go-To-The-Middle* strategy and proceed with the analysis of its performance. This strategy is solving the relaying problem on planar terrain without obstacles, using the simple type of robots described earlier. In the *Look*-operation a relay station v_i observes the positions $p_t(i+1)$ and $p_t(i-1)$ of its neighbors. In the *Compute*-operation it calculates the point p lying directly in the middle between $p_t(i+1)$ and $p_t(i-1)$ and eventually moves to that point during the last *Move*-operation. One may be worried that in this case the movement distance per step is bounded from above only by $\mathcal{R}$ for relay stations. In general this is true, nevertheless in our scenario one can show that the maximum movement distance per time step is bounded by 1, as the explorer never moves more per time step.

Let $d_t = |b - p_t(0)|$ denote the distance of the explorer to the base station. In the following part of the paper we will prove that the following theorem holds.

THEOREM 1. *There exists a movement pattern for the explorer such that the Go-To-The-Middle strategy forces the explorer to slow down to a speed of $\mathcal{O}(1/d_t)$.*

The explorer is obliged to keep an appropriate number of relay stations within the chain – this means that it has to insert new relay stations into the chain (changing the value n_t) when it moves outbound of the base station and retrieve stations from the chain when it moves inbound. To allow for a proper functioning of the communication chain we assume that the explorer keeps $c \cdot d_t/\mathcal{R}$ relay stations in the chain, with an appropriately chosen small constant c. For an interested reader, the *Go-To-The-Middle* strategy is explained in more detail in [8].

This definition implies that the number of relay stations within the chain is always only by a constant c worse than the optimal number of stations. Though, the movement of the explorer may be severly hindered by the relay stations – if v_1 keeps up after the explorer only very slowly, the explorer has to reduce its speed appropriately (in order not to exceed the distance limit of $\mathcal{R}$ to v_1). In the rest of this section we will be investigating how much the speed of the explorer can drop due to this effect.

Imagine the line L_t connecting the base station and the explorer in time step t. We define the vector u_t as the distance of relay station v_i to the line L_t at the beginning of time step t (see Fig. 3). Furthermore, let $\tilde{p}_t(i)$ be the projection of $p_t(i)$ onto the line L_t. Clearly, $|p_t(i) - \tilde{p}_t(i)|_2 = u_t[i]$. As both the base station and the explorer lie on the line L_t, we have $u_t[1] = u_t[n_t+1] = 0$. Observe, that for two stations v_i and v_{i+1} the condition $|p_t(i) - p_t(i+1)|_2 \leq \mathcal{R}$ implies $|u_t[i] - u_t[i+1]| \leq \mathcal{R}$. For a vector w we will denote by $|w|$ the sum of all of its elements. By the former

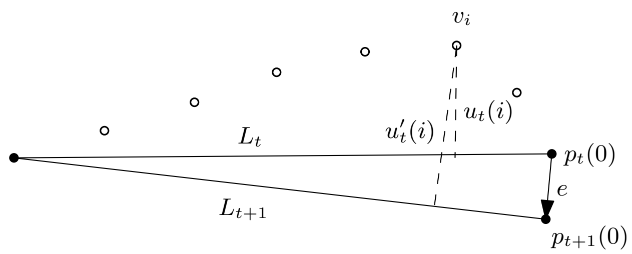

Figure 3: Explorer's movement and lines L_t, L_{t+1}

observation we have

$$|u_t| = \sum_{i=0}^{n_t+1} u_t[i] \leq 2 \cdot \sum_{i=1}^{\lceil n_t/2 \rceil} i \cdot \mathcal{R} \leq \mathcal{R} \cdot n_t^2 \ . \qquad (1)$$

In the following discussion we will index vector items from $0, \ldots, m+1$ for a vector of size $m+2$. We define the function $g : \mathbb{R}^{m+2} \to \mathbb{R}^{m+2}$, such that for a vector $w \in \mathbb{R}^{m+2}$

$$(g(w))[i] = \begin{cases} 0 & \text{if } i = 0 \\ \frac{1}{2}(w(i-1) + w(i+1)) & \forall i \in [1, \ldots, m] \\ 0 & \text{if } i = m+1 \end{cases}$$

The following property of the *Go-To-The-Middle* strategy is discussed in more detail in [8], we only cite it here.

FACT 2. *Let vector w describe the distances of relay stations to a line L. Then after applying one step of the Go-To-The-Middle strategy by all relay stations, the distances to line L are given by $g(w)$.*

The next technical lemma shows a straightforward observation which will be important for our following investigation.

LEMMA 3. *For any vector $w \in \mathbb{R}^{m+1}$ it holds*

$$|w| - |g(w)| = \frac{w[1] + w[m]}{2} \ .$$

PROOF. Note that for any $i \in [2, \ldots, m-1]$ the value $w[i]$ contributes $w[i]/2$ to $(g(w))[i-1]$ and the same amount to $(g(w))[i+1]$. That means, that the sum $\sum_{i=2}^{m-1} w[i]$ is completely included in $|g(w)|$. On the other hand, the values $w[1]$ and $w[m]$ only contribute respectively $w[1]/2$ to $(g(w))[2]$ and $w[m]/2$ to $(g(w))[m-1]$. So, $|g(w)| \leq |w|$ and the only part of the sum $|w|$ missing in $|g(w)|$ is $w[1]/2 + w[m]/2$. ☐

Explorer's movement. In the following we will be only dealing with explorer's movement on a circle around the base station. For simplicity of the argumentation we assume that all stations are on one side of the line L_t. If this is guaranteed at the beginning and the explorer moves in one direction then this invariant is maintained. Without loss of generality we may assume that in a step the explorer moves first and then all relay stations apply the *Go-To-The-Middle* strategy. By u'_t we will denote the vector of distances of relay stations to the line L_{t+1} *prior* to applying the *Go-To-The-Middle* strategy. Then $g(u'_t) = u_{t+1}$. We first introduce a technical lemma and then formally show that $u'_t[i] - u_t[i]$ is proportional to the distance of the relay station v_i to the base station and the length explorer's movement vector in time step t.

LEMMA 4. *For time step t it holds*

$$(n_t/c - i) \cdot \mathcal{R} \leq |b - \tilde{p}_t(i)|_2 \leq (n_t - i + 1) \cdot \mathcal{R} \; .$$

PROOF. If the point $\tilde{p}_t(i)$ was at a distance larger than $(n_t - i + 1) \cdot \mathcal{R}$ from the base station, then the same would hold for point $p_t(i)$. This cannnot happen, since then at least one pair of relay stations in the chain between v_i and the base would exceed the allowed maximum distance $\mathcal{R}$.

We can apply the same idea to the distance between $\tilde{p}_t(i)$ and the explorer: we obtain a lower bound on $|b - \tilde{p}_t(i)|_2 \geq d_t - i \cdot \mathcal{R}$. Since $n_t = c \cdot d_t/R$ we have $|b - \tilde{p}_t(i)|_2 \geq (n_t/c - i) \cdot \mathcal{R}$. $\square$

LEMMA 5. *Let the length of the explorer's movement vector be e in some time step t, and assume that it holds $|d_t| = |d_{t+1}|$. Then*

$$\begin{aligned} \frac{(n_t - i \cdot c + c) \cdot e}{n_t} - \mathcal{O}(e/n_t) &\leq u'_t[i] - u_t[i] \\ &\leq \frac{(n_t - i \cdot c + c) \cdot e}{n_t} \; . \end{aligned}$$

PROOF. Denote by δ the distance between the base station and the projection $\tilde{p}_t(i)$. Let both angles α and β be routed at b. Angle α is spanned between the lines connecting b to $p_t(i)$ and $p_t(0)$. Angle β is the angle between L_t and L_{t+1}. From obvious geometrical observations we can give bounds on the cosine and sine of both angles. In order to bound $u'_t[i] - u_t[i]$ from below we use the trigonometrical bound on the angles. The technical proof of the following equation can be found in the appendix.

$$\begin{aligned} u'_t[i] &= \sqrt{u_t^2 + \delta^2} \cdot \sin(\alpha + \beta) \\ &= u_t[i] + \frac{\delta \cdot e}{d_t} \cdot \sqrt{1 - \frac{e^2}{d_t^2}} - u_t[i] \cdot \frac{e^2}{2 \cdot d_t^2} \; . \end{aligned} \quad (2)$$

We can bound $u_t[i] \leq n_t \cdot \mathcal{R}$ and then

$$u_t[i] \cdot \frac{e^2}{2 \cdot d_t^2} \leq \frac{c \cdot e^2}{n_t} \leq \mathcal{O}(e/n_t) \; ,$$

since $d_t = \mathcal{R} \cdot n_t/c$. Using this and by Lemma 4 we obtain

$$\begin{aligned} u'_t[i] &\geq u_t[i] + \frac{(n_t/c - i) \cdot \mathcal{R} \cdot e}{\mathcal{R} \cdot n_t/c} - \mathcal{O}(e/n_t) \\ &\geq u_t[i] + \frac{(n_t/c - i) \cdot e \cdot c}{n_t} - \mathcal{O}(e/n_t) \; . \end{aligned}$$

The bound from above follows easily

$$\begin{aligned} u'_t[i] &= u_t[i] + \frac{\delta \cdot e}{d_t} \sqrt{1 - \frac{e^2}{d_t^2}} - u_t[i] \frac{e^2}{2 \cdot d_t^2} \\ &\geq u_t[i] + \frac{\delta \cdot e}{d_t} \geq \frac{(n_t - i + 1) \cdot c}{n_t} \; . \end{aligned}$$

$\square$

PROOF OF THEOREM 1. By Lemma 5 a movement of the explorer by e around the base station increases $|u_t|$ by at least

$$\begin{aligned} |u'_t| - |u_t| &\geq \sum_{j=1}^{n_t} u'_t[j] - u_t[j] \geq \sum_{j=1}^{n_t/c} u'_t[j] - u_t[j] \\ &\geq \sum_{j=1}^{n_t/c} \frac{(n_t - i \cdot c + c) \cdot e}{n_t} - \mathcal{O}(e/n_t) \\ &\geq \frac{e}{n_t} \cdot \frac{n_t^2}{2c} - \mathcal{O}(e) \geq \Omega(n_t \cdot e) \; . \end{aligned}$$

The first inequality holds because $u'_t > u_t$. On the other hand, by Lemma 3 applying *Go-To-The-Middle* to u'_t reduces it by at most $\mathcal{R}$, as it must hold $u'_t[2] \leq \mathcal{R}$ and $u'_t[n_t - 1] \leq \mathcal{R}$. This implies that on average the change $|u'_t| - |u_t|$ may not exceed $\mathcal{R}$. Otherwise the value of $|U|$ would increase with time to infinity, violating the condition expressed in Eq.(1). That implies $\Omega(n_t \cdot e) \leq \mathcal{R}$ and so $e \leq \mathcal{O}(1/n_t)$. $\square$

Using the technique developed above, we can argue on a lower bound on the speed of the explorer, when it preserves the distance to the base station. Note that every time when *Go-To-The-Middle* is applied, the value $|u'_t|$ is decreased by at least $u'_t[1]/2$. If at the beginning of each time step the station v_1 remains in some bounded distance $r < R$ to the explorer, then we have $u'_t[1]/2 > \Omega(r)$ from an obvious geometric argument. Assume that this holds. For any movement of the explorer by e it holds by Lemma 5

$$\begin{aligned} |u'_t| - |u_t| &\leq \sum_{j=1}^{n_t} u'_t[j] - u_t[j] \leq \sum_{j=1}^{n_t} \frac{(n_t - j \cdot c + c) \cdot e}{n_t} \\ &\leq \mathcal{O}(n_t \cdot e) \end{aligned}$$

Therefore the explorer must move by at least $\Omega(1/(n_t \cdot r)$ on average to balance out the decrease in $|u|$ due to the application of *Go-To-The-Middle*.

The argumentation depends on the fact, that there exists a constant $r < R$ such that $|p_{t+1}(1) - p_t(0)|_2 \leq r$. Experimental evaluations of various movement patterns of an explorer around the circle confirm that there exists such a constant r. Though, this does not consitute a proof, it indicates that the provided bounds are tight, i.e. that the *Go-To-The-Middle* strategy is able to maintain a speed of at least $\Omega(1/d)$ for an explorer moving in distance d to the base station.

3. CHASE-EXPLORER STRATEGY

In this section we describe the *Chase-Explorer* strategy and provide its analysis for the obstacle-free scenario.

Chase-Explorer works as follows: at the beginning of a time step t, relay station v_i looks at the position of the station v_{i-1} which is $p_t(i-1)$. Relay station v_i computes the coordinates of a point which is on the interval connecting $p_t(i-1)$ and the base station, and in distance $R := \mathcal{R} - 1$ from $p_t(i-1)$. The positions are depicted in Fig. 4. Informally speaking, the relay stations try to keep as near to the direct line connecting the base station and the explorer as possible. At the same time they try to maintain a distance of R to the previous station in the chain. This is different to *Go-To-The-Middle*, where stations place themselves according to the positions of their neighbors only. All relay

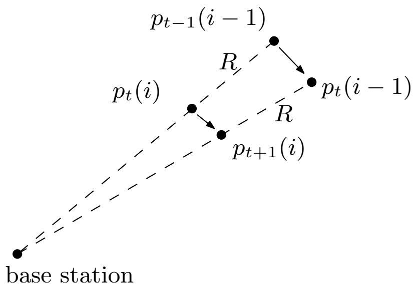

Figure 4: The *Chase-Explorer* strategy

stations move simultaneously, thus stations have to precompute their new positions during the *Look* and *Compute* parts of each step.

The movement of the explorer can change its distance to the base station. Thus, it may be necessary to change the number of relay stations in the chain. The base station decides to insert a new relay station when the last relay station v_{n_t} reaches a distance larger than R to the base station at the end of time step $t-1$. The new relay station is inserted on the line between the position $p_t(n_t)$ and the base station, keeping a distance of R to the position $p_t(n_t)$. On the contrary, if the last relay station comes too near to the base station, it is removed.

Chase-Explorer ensures by its design that the explorer is able to move freely, without being hindered by the chain. To ensure correctness of the *Chase-Explorer* strategy it is necessary to show that the relay stations will be able to chase the explorer without exceeding their maximum speed and that the distance limit of $\mathcal{R}$ won't be exceeded between neighbored stations.

THEOREM 6. *Assume that relay stations know the precise location of the base station relative to their own position. Then for the Chase-Explorer strategy and an explorer with maximum speed* 1 *it holds*

- *the speed of the relay stations won't exceed* 1*,*
- *the distance between neighbors in the chain never exceeds* $\mathcal{R}$.

After showing the correctness of the *Chase-Explorer* strategy we will analyze the behavior of the chain in presence of imprecise localization methods, where stations only know an approximation of the position of the base station, with an appropriately bounded additive error.

THEOREM 7. *Let* $\gamma_t(i)$ *be the hop-distance of* i *to the base station, i.e,* $\gamma_t(i) := n_t - i + 1$. *If station* v_i *knows the position of the base station relative to its own position with additive error at most* $\frac{R}{25} \cdot \gamma_t(i)$ *then*

$$n_t \leq 1.5 \cdot \frac{d_t}{\mathcal{R}-1} + 1 \ .$$

The following theorem shows limits for the additive error.

THEOREM 8. *If the additive error is larger than R, then there is a route of the explorer and a way to fix the localization errors, so that*

$$\lim_{t\to\infty} n_t \to \infty$$

while d_t *remains constant.*

3.1 Correctness

We first investigate the correctness of the *Chase-Explorer* strategy assuming no localization errors.

PROOF OF THEOREM 6. First we want to show that no relay station is required to move for a distance larger than 1 during one time step, providing that the explorer does not exceed its maximal speed of 1 per time step. Assume that station v_{i-1} moves a distance of at most 1 every time step. A movement of station v_{i-1} is depicted on Fig. 4 between points $p_{t-1}(i-1)$ and $p_t(i-1)$. The distances $|p_t(i) - p_{t-1}(i-1)|_2 = |p_{t+1}(i) - p_t(i-1)|_2$ are both equal to R. By an obvious geometric argument the distance between $p_t(i)$ and $p_{t+1}(i)$ traveled by station v_i in time step t is not greater than the distance between $p_{t-1}(i-1)$ and $p_t(i-1)$. So, the movement distance of station v_i is bounded by 1 if station v_{i-1} has moved by at most 1.

We consider the distance between a station v_i and its neighbor v_{i-1} at the end of time step t. Obviously the distance between $p_t(i)$ and $p_{t-1}(i-1)$ is exactly R after time step t. Since the station v_{i-1} can move for a distance of at most 1 during the time step t, the distance between $p_{t-1}(i-1)$ and $p_t(i-1)$ is at most 1. Thus, by the triangle inequality, the distance between $p_t(i)$ and $p_t(i-1)$ is at most $R+1$. This assures that the distance between the relay station v_i and the explorer is at most $R+1 \leq \mathcal{R}$ at the beginning of each time step. □

Due to space considerations, we won't argue about the local correctness of the *Chase-Explorer* strategy when only imprecise approximations of the neighbor's and base station's position are known to the relay stations. Instead, we turn our attention to the more interesting question on how the chain behaves globally if stations have only an imprecise approximation of their global position.

3.2 Unprecise Base Station Localization

In the following, we will consider the behavior of the *Chase-Explorer* strategy when the relay stations know only an imprecise estimate of the base station's position relatively to their own position. Formally, we denote the position of the base station known by station v_i as $b_t(i)$ at time step t, whereas b is the real base station's position. In the remaining part we will denote the interval or line defined by two points x, y as $\langle x, y \rangle$.

It is not hard to imagine that the strategy is able to work with a localization scheme with some bounded additive error ϵ (i.e. the GPS, so that $|b - b_t(i)|_2 \leq \epsilon$). We though want to show that much weaker localization scheme is enough for *Chase-Explorer* to work properly and to attain a reasonable performance (in terms of the number of relay stations used). Recall, that $\gamma_t(i) := n_t - i + 1$. We aim at showing that a localization system such that $|b_t(i) - b|_2 \leq \epsilon \cdot R \cdot \gamma_t(i)$ is enough. This means, that stations which are further apart from the base station are allowed to have a larger error in their localization. Assume that the value n_t always changes at the beginning of a time step. Therefore for during step t, such that $n_t = n_{t-1} + 1$ we already have an increased value of $\gamma_t(i)$ for station i.

This weak accuracy brings some problems: since the accuracy depends on the number $\gamma_t(i)$ and the number $\gamma_t(i)$ depends on the accuracy (due to weak accuracy the chain does not follow an optimal line and additional relay stations must be introduced) one might be worried that the chain gets infinitely long, with the accuracy getting weaker parallely. Theorem 7 shows that this behavior does not occur if ϵ is bounded sufficiently. Theorem 8 shows the contrary, i.e. that a localization system with ϵ large can lead to unstable behavior.

Proof of Theorem 7. Let $r_t(i) := |b - p_t(i)|_2$ be the distance of the station v_i to the base station in time step t. Then we define

$$\begin{aligned} u_t(i) &:= |r_{t-1}(i) - r_t(i+1)| \\ \tilde{u}_t(i) &:= |r_t(i) - r_t(i+1)| \leq u_t(i) + 1 \end{aligned}$$

Let $\alpha_{t-1}(i)$ be the angle at $p_{t-1}(i)$, between the intervals $\langle b, p_{t-1}(i)\rangle$ and $\langle b_t(i+1), p_{t-1}(i)\rangle$. With other words $\alpha_{t-1}(i)$ is the angle, measured at $p_t(i)$ between the real position of the base station and the approximation of the base station's position known by v_i.

Let $|b - b_t(i+1)|_2 \leq \epsilon_t(i+1) = \epsilon \cdot R \cdot \gamma_t(i)$. We introduce three lemmas, which relate the values of u, α, r to each other. Though short versions of the proof of these lemmas can be found in the appendix, we refer the interested reader to the full version of the paper.

Lemma 9. *For every i and t such that $\gamma_t(i) > 2$*

$$u_t(i) \geq R \cdot (\cos(\alpha_{t-1}(i)) - \sin(\alpha_{t-1}(i))) \ .$$

Lemma 10. *For every $\alpha_{t-1}(i)$*

$$\begin{aligned} \cos \alpha_{t-1}(i) &\geq \sqrt{1 - \frac{\epsilon_t^2(i+1)}{r_{t-1}^2(i)}} \ , \\ \sin \alpha_{t-1}(i) &\leq \frac{\epsilon_t(i+1)}{r_{t-1}(i) - \epsilon_t(i+1)} \ . \end{aligned}$$

Lemma 11. *Let $u \leq u_{t-1}(i)$ and $u \geq 3$ for all t and i such that $i \leq n_t - 2$. Then for all t and i such that $i \leq n_t - 2$ it holds*

$$r_{t-1}(i) \geq \frac{1}{3} \cdot \gamma_t(i+1) \cdot u$$

Take any time step $t-1$. Let $u = 0.7 \cdot R$. We want to show that $u_t(i) \geq u$ for all i such that $\gamma_t(i) \geq 1$ if $u \leq u_{t-1}(i)$ and $R \geq 5$. For the station standing next to the base station we assume that it may be at any distance to the base station, therefore $u_t(n_t - 1)$ may be as low as 0.

Bringing together Lemma 9, 10, and 11 we obtain the following lower bound

$$u_t(i) \geq R \cdot \left(\sqrt{1 - \frac{9 \cdot \epsilon^2 \cdot R^2}{u^2}} - \frac{\epsilon \cdot R}{1/3 \cdot u - \epsilon \cdot R} \right) \ .$$

To prove our claim it should hold

$$R \cdot \left(\sqrt{1 - \frac{9 \cdot \epsilon^2 \cdot R^2}{u^2}} - \frac{\epsilon \cdot R}{1/3 \cdot u - \epsilon \cdot R} \right) \geq u \ .$$

Plugging in $u = 0.7 \cdot R$ the above holds for all $\epsilon \leq 1/25$. Therefore we have $u_t(i) \geq 0.7 \cdot R$ for all i and t such that $\gamma_t(i) \geq 1$.

For the sake of contradiction assume now that there is a time step such that the number of relay stations used by *Chase-Explorer* exceeds $1.5\lceil d_t/R \rceil + 1$ (recall that d_t is the distance of the explorer to the base station). By our previous considerations this would mean that $d_t \geq 0.7 \cdot R\,(1.5\lceil d_t/R \rceil + 1) \geq 1.05 \cdot d_t > d_t$, which is clearly a contradiction. □

Proof of Theorem 8. The distance of relay station v_i to the base station clearly cannot exceed $R \cdot \gamma_t(i)$. So, if we have $\epsilon > R$ the adversary can select a position $b_t(i)$ all around station v_i. This allows the adversary to completely control the shape of the chain. More specifically, it can create an infinitely long chain with stations such that $r_t(i) \geq r_t(i-1)$. This part of the chain increases its distance to the relay stations instead decreasing it. This shows that for $\epsilon > R$ the chain may increase to an infinite length, while the explorer remains in the same distance to the base station. □

3.3 Localization Scheme

We design a localization scheme such that *Chase-Explorer* can work without any GPS-like system. The only requirement is that stations can measure the positions of their local neighbors with some precision ϵ. Then we can design a localization scheme with an additive error of

$$E := \frac{\epsilon}{1 - 1/(0.7 \cdot R)} \cdot \gamma_t(i) \ .$$

For simplicity, let us fix the position of the base station b to the center of the coordinate system. Let $\tilde{p}_t(i)$ be the position of station v_i, as v_i approximates it. Note that this notation is equivalent to that used previously, where a station knew an approximation of the base station's position $b_t(i)$ and its own exact position. Here a relay station knows the exact position of the base station but only an approximation of its own position.

The scheme works as follows. Relay station v_i transmits in time step t the value of $\tilde{p}_t(i)$ to v_{i-1} in a communication message. Station v_{i-1} can then measure its position relatively to v_i and compute $\tilde{p}_{t+1}(i-1)$. If $|\tilde{p}_t(i) - p_t(i)|_2 < j \cdot \epsilon$ and the local measurement has an error of ϵ, then

$$|\tilde{p}_{t+1}(i-1) - p_{t+1}(i-1)|_2 < (j+1) \cdot \epsilon \ .$$

This scheme works continously, so that a station has a new approximation of its position in each time step.

Base station v_i's approximation in time step t depends on a message which went j hops from the base station to v_i. Unfortunately j must not be necessarily equal to $\gamma_t(i) - 1$. During the time when the message traveled from the base station to v_i several relay stations might have been removed. On the other hand, by Theorem 7 it holds $u_t(i) \geq 0.7 \cdot R$ and so in j time steps only $j/(0.7 \cdot R)$ relay stations might have been removed. Thus, $j - j/0.7 \cdot R \leq \gamma_t(i)$ and so the error of the localization scheme is at most E.

Note that this scheme guarantees that the explorer will never get lost in terrain, even if there is no infrastructure for localization (GPS) and its own odometry cannot be relied on. Thus, the communication chain can be even used for having some (although imprecise) localization.

4. TERRAIN WITH OBSTACLES

Let us recall the model we are dealing with. We are concerned with a plane with obstacles represented by points. We are aiming at maintaining a chain of relay stations which is as short as possible but also topologically equivalent to the path traveled by the explorer. We are using communication between relays to coordinate their actions. We though assume a very weak communication model: messages may travel only one hop per time step, which means that the messages have a propagation speed proportional to the movement speed of stations.

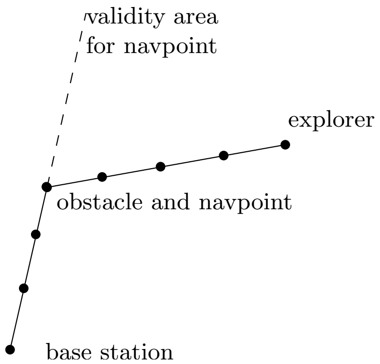

Figure 5: Navpoint and its validity area

We will denote the movement sequence executed by the explorer as σ. The *Chase-Explorer* strategy will be able to put the explorer into wait mode, in which it stops executing σ and waits for *Chase-Explorer* to finish some maintenance. As *Chase-Explorer* is deterministic, we allow the adversary to generate $\sigma[t]$ basing on the behavior of *Chase-Explorer* for $\sigma[1], \ldots, \sigma[t-1]$.

The basic idea for *Chase-Explorer* is that relay stations setup navigation points (or navpoints for short) on the position of obstacles. These work as a base station for the part of the chain which lies behind them (in the direction of the explorer). Such a situation is depicted on Fig. 5. It is natural for the stations behind the obstacle to assume that the navpoint is the base station appropriate for them as long as the explorer stays in the validity area. Relay stations employed in the chain use the standard *Chase-Explorer* strategy, while exchanging messages about new navpoints and changing the coordinates of the reference navpoint appropriately.

Unfortunately, even if each station passes the notice about the creation of a new navpoint as fast as possible, the chain is brought out of order. This is the case, since during the time which passed between establishing the navpoint and receiving information about this fact, a station v_i performs its movements aligning to its old reference navpoint. This can cause the chain to become unnecessarily longer. Furthermore, during this time relay stations are heading in a wrong direction and may hit an obstacle that they usually won't hit. This causes new navpoints to be established, the chain being extended even more — we have to ensure that the chain length nevertheless stays bounded.

The main result of the analysis of *Chase-Explorer* for a terrain with obstacles is presented in Theorem 12. For its understanding we first have to introduce some notation.

4.1 Execution in Epochs

The navpoint which station v_i uses as its base station will be called v_i's reference navpoint. Wherever this does not cause any ambiguity, we will treat the explorer and the base station as navpoints. By δ_t we will denote the number of stations employed in the chain in time step t.

We will partition the execution of the input sequence into epochs. Each epoch will consist of a certain number of steps of execution of the input sequence σ (normal phase), followed by some wait steps (maintenance phase). Wherever it is clear from context which epoch is meant, we will denote $\delta = \delta_t$ where t is the first time step of epoch E_q. The normal phase of an epoch has duration δ. During the normal phase we won't care much about the shape of the chain around obstacles – the relay stations will do what they can to minimize the length of the chain, but we will allow for some losses here. In the maintenance phase we will be bringing the chain to an optimal order. Whenever epoch E_q has ended, the next epoch E_{q+1} starts afterwards in the next time step.

4.2 Navpoints and Terrain Complexity

We assume that stations are able to detect obstacles and setup navpoints where the chain hits an obstacle. The chain may hit an obstacle during its movement in the middle of a time step – for simplicity of description we assume that the station is able to detect that situation when it occured a step, setup the navpoint and start to behave accordingly with regard to the navpoint for the rest of the time step. Note, that this is not a violation of the LCM-model, since a station may have recognized the obstacle already at the beginning of the step and planned its movement according to the projected time of hitting the obstacle. After a time step t a navpoint setup by v_i will be always on the interval $\langle p_t(i-1), p_{t+1}(i)\rangle$. The station v_i which established a navpoint is responsible for maintaining it first, i.e. remembering the position of the navpoint. Station v_i may pass the information associated with the navpoint to v_{i-1} or v_{i+1} if one of these stations becomes responsible for maintaining the navpoint. There are a few rules governing the behavior of relay stations located next to the navpoint.

- Station v_i maintaining navpoint s positions itself in time step t on the interval $\langle s, s_r\rangle$, where s_r is the current reference navpoint of v_j, in distance $R - |s - p_t(i-1)|_2$. This means, that the station is ensuring that it holds $|s - p_{t+1}(i)|_2 + |s - p_t(i-1)|_2 = R$. By the triangle inequality $|p_{t+1}(i) - p_t(i-1)|_2 \leq R$.

- When station v_i maintains navpoint s and it happens that $|s - p_t(i-1)|_2 > R$ then station v_i passes the responsibility for maintaining s_j to v_{i+1}, updates its reference navpoint to s and starts to follow v_{i-1} in the usual way. This may be imagined as station v_i passing the navpoint.

- If station v_i maintains navpoint s and it holds $|p_t(i-2) - s|_2 \leq R$, then the responsibility for maintaining s is passed over to v_{i-1} in time step t. Furthermore, station v_{i-1} updates its reference navpoint to that used by v_i. In that case v_{i-1} is passing the navpoint.

The terrain complexity for each epoch depends on the number of obstacles in the activity area of the explorer in this epoch. Let $A_t(s)$ be the area defined by the triangle $s, p_t(0), p_{t+1}(0)$. Let N_q denote the normal phase of epoch E_q. Then we define $A_q(s) = \bigcup_{t \in N_q} A_t(s)$. Let $S = s_1, \ldots, s_k$ be the navpoints present at the beginning of E_q, whereas s_k is the base station. Let s_i be the navpoint with smallest ID, such that all navpoints $s_1, \ldots, s_{i-1}$ are in its validity area (the validity area of a navpoint is shown in Fig. 5) and the explorer has not moved outside of its validity area during N_q. Then k_q denotes the number of obstacles within $\bigcup_{s \in \{s_1, \ldots, s_{i-1}\}} A_q(s)$.

Now we are ready to state our main result.

THEOREM 12. *Assume that the explorer was allowed to move δ time steps during an epoch. Then the number of waiting time steps in this epoch is $\mathcal{O}(R \cdot \delta \cdot k_q)$. For any time step in an epoch, the number of relay stations used within the chain is at most* $(1 + 1/R) \cdot \delta$.

A chain layout C is defined by the sequence of navpoints it uses and by the orientation of the chain around the corresponding obstacles. Formally, a layout is defined by a sequence (s_i, z_i) where s_i describes the navpoint and $z_i =$ right if the chain passes the obstacle s_i on the right side of the line $\langle s_{i+1}, s_i \rangle$ and $z_i =$ left otherwise. Navpoints are indexed in the same direction as the relay stations, so that s_0 defines the explorer and s_k the base station. Two layouts C and C' are topologically equivalent under a set of obstacles S if starting at layout C the relay stations employed in the chain can move their positions so that they form layout C' without having to go over any obstacle from S (which we do not allow generally). If S is the complete set of obstacles, then the two layouts are topologically equivalent.

When a navpoint is established by a station, no further notice about this fact is issued to other stations. All stations behind the navpoint still use their old reference navpoint, only those which directly pass the navpoint start using it as their reference navpoint as described earlier. This may appear counterproductive now – we might have as well sent a message which notifies stations about the navpoint – but it keeps the description of the strategy clean and concise and does not cause any significant loss in performance. It has to be noted, that the chain still remains connected if stations use different (and possibly wrong) navpoints. It just makes the chain longer than necessary.

4.3 Maintenance Phase

During the normal phase our strategy has pretty well ignored obstacles. It only considered them as much as to stay connected. During the maintenance phase we have to bring the chain back to order. We aim at achieving an optimal chain layout at the end of the maintenance phase. The base station is responsible for guiding the general layout of the chain in the maintenance phase. For this purpose it collects information about obstacles in relevant areas of the terrain and recomputes an optimal layout of the chain, basing on the partial information about obstacles already known. During the reorganization of the chain basing on the specifications by the base station new obstacles may be encountered which were not known before. This new encounters are communicated to the base station, which is then able to reorganize the chain according to new information.

Denote the layout of the chain at the beginning of the maintainance phase by C. Assume first that the base station knows the positions of all obstacles on the terrain. Then the base station has the possibility to compute a layout for the chain, which is topologically equivalent to C and which has the shortest possible length. Unfortunately, the base station does not have this full information about the obstacles available. At the beginning of a maintenance phase we may only assume that it knows obstacle positions from previous epochs and the obstacles which caused the chain to setup navpoints in the current epoch. Basing on this partial information, the base station may also compute a layout C' for the chain, which is topologically equivalent to C under the set of known obstacles (i.e. C and C' are topologically equivalent only if there are no obstacles on the way between C and C'). Otherwise, C' is infeasible. In this situaion encountering a new obstacle provides the base station with new information about the terrain and it update its layout.

We neglect here the question on how the base station can compute the new layout. This question is treated in more detail in the full version of the paper.

The chain of relay stations obtains information about a new layout from the base station. It has to reorganize by letting each station update its reference navpoint and align itself w.r.t. this reference navpoint to its predecessor. This is performed starting at the explorer and proceeding in the direction of the base station. If a station has to realign, it moves from one position to another on a circle with diameter $2R$, therefore requiring at most $2R$ time steps for this operation. Thus, the reorganization of the chain requires $2R \cdot (1 + 1/R) \cdot \delta$ time steps.

If the reorganization of the chain fails as new obstacles turn out on the way, the base station has to compute a new layout and a new reorganization is performed. This must be repeated at most k_q times, since afterwards the base station knows all obstacles which the chain can encounter. Therefore the maintenance phase takes at most $\mathcal{O}(R \cdot \delta \cdot k_q)$ time steps.

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APPENDIX

A. PROOFS OF TECHNICAL LEMMAS

PROOF OF EQ.(2). It holds

$$\begin{aligned}
\cos\alpha &= 1 - \frac{e^2}{2\cdot d_t^2} \\
\sin\alpha &= \sqrt{\frac{e^2}{d_t^2} - \frac{e^4}{d_t^4}} \approx \frac{e}{d_t} \\
\cos\beta &= \frac{\delta}{\sqrt{u_t^2+\delta^2}} \\
\sin\beta &= \frac{u_t}{\sqrt{u_t^2+\delta^2}} \ .
\end{aligned}$$

Then by applying standard trigonometric inequalities we obtain

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
&= \frac{e}{d_t}\cdot\frac{\delta}{\sqrt{u_t^2+\delta^2}} + \\
&\quad \left(1-\frac{e^2}{2\cdot d_t^2}\right)\cdot\frac{u_t}{\sqrt{u_t^2+\delta^2}}
\end{aligned}$$

and furthermore

$$\sqrt{u_t^2+\delta^2}\,\sin(\alpha+\beta) = \frac{\delta\cdot e}{d_t} + \left(1-\frac{e^2}{2\cdot d_t^2}\right)\cdot u_t \ .$$

□

PROOF OF LEMMA 9. In order to simplify notation let us set $r := r_t(i+1)$, $u := u_t(i)$, $\alpha := \alpha_{t-1}(i)$. From the Law of Cosines we have

$$r^2 = (r+u)^2 + R^2 - 2(r+u)\cdot R\cdot\cos\alpha \ .$$

Looking at the above as on a polynomial of u yields the following positive root

$$u = R\cdot\cos\alpha + \sqrt{r^2 - R^2\sin^2\alpha} - r \geq R\cdot(\cos\alpha - \sin\alpha) \ .$$

The inequality comes from the fact that $\sqrt{r^2 - c^2\sin^2\alpha} - r$ is a non-decreasing function of r. Since $r \geq R\cdot\sin\alpha$ for the triangle to exist, we can plugin $r = R\cdot\sin\alpha$ and obtain the requested lower bound. □

PROOF OF LEMMA 10. Let us consider the triangle between b, $b_t(i+1)$ and $p_{t-1}(i)$. Obviously the edge $\langle b, b_t(i+1)$ has length $\epsilon_t(i+1)$. Stating $\cos\alpha$ as a function of $x = |b_t(i+1) - p_{t-1}(i)|_2$ one obtains

$$\cos\alpha = \frac{x^2 + r_{t-1}(i)^2 - \epsilon_t(i+1)^2}{2x\cdot r_{t-1}(i)} \ . \tag{3}$$

This function obtains its minimum for

$$x = \sqrt{r_{t-1}(i)^2 - \epsilon_t(i+1)^2} \ .$$

Plugging in this x into Eq. (3) yields the lower bound. Using the same triangle one obtains the upper bound on

$$\sin\alpha_{t-1}(i) = \frac{\epsilon_t(i+1)}{x} \leq \frac{\epsilon_t(i+1)}{r_{t-1}(i) - \epsilon_t(i+1)} \ .$$

□

PROOF OF LEMMA 11. Observe that the distance $r_{t-1}(i)$ is equal to a sum of $\tilde{u}_{t-1}$

$$r_{t-1}(i) = \sum_{j=i}^{n_{t-1}} \left(r_{t-1}(j) - r_{t-1}(j+1)\right) = \sum_{j=i}^{n_{t-1}} \tilde{u}_{t-1}(j) \ .$$

Observe that $n_{t-1} \geq n_t - 1$, since the number of relay stations in the chain can change by at most 1 during one time step. Then we have

$$\begin{aligned}
r_{t-1}(i) &\geq \sum_{j=i}^{n_{t-1}} (u_{t-1}(j) - 1) \\
&\geq (n_{t-1} - i) \min_{j\in\langle i, n_{t-1}-1\rangle} u_{t-1}(j) - 1 \\
&\geq (n_t - i - 1)\cdot(u-1) \ .
\end{aligned}$$

Note that we have lower bounded $u_{t-1}(n_{t-1})$ by 0, since the last relay station can be very near to the base station. Since $u \geq 3$ it follows

$$r_{t-1}(i) \geq (n_t - i - 1)(u-1) \geq \frac{1}{3}\cdot\gamma_t(i+1)\cdot u \ .$$

□