

1 Wireless Overlay Networks

Wireless communication networks are often modeled in terms of n points in the plane. Each point corresponds to a wireless communication device that we refer to as a node of the network. In principle, every node can directly communicate with every other node if messages are transmitted with sufficient energy. However, even in an ideal setting, the energy requirement for sending a message is proportional to the squared distance between sender and receiver. Thus, it might be possible to save energy and reduce interferences by sending messages through intermediate nodes. One way to design routing algorithms that use intermediate nodes is to use a sparse overlay network in which each node is connected only with a few neighbors and messages are sent along the virtual edges of the overlay network. Such an overlay network should achieve the following informal goals:

- In order to save energy and to reduce interferences, nodes should only be connected to nodes that are placed relatively close.
- The overlay network should support a simple, local-control routing algorithm such that each node can locally decide along which outgoing edge a message with a given destination should be forwarded.
- It should be possible to establish the overlay network in a distributed fashion with very small communication overhead.

The last requirement is important as wireless networks are usually highly dynamic in the sense that nodes enter and leave the system and nodes also might change their coordinates. Hence, we seek for overlay structures that can be built almost instantaneously with very small communication overhead. Such overlay networks are often referred to as *ad hoc networks*.

1.1 The communication model

We use the following model for wireless networks:

- Nodes (wireless communication devices) correspond points in Euclidian space \mathbb{R}^2 .
- Euclidian distance: Two nodes $u = (u_1, u_2)$ and $v = (v_1, v_2)$ have distance

$$||uv|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} .$$

- Energy consumption for direct communication: Ideally, the energy consumption for a transmission between two nodes u and v is $||uv||^2$. A more realistic estimation takes into account that the signal may be disturbed by obstacles like trees, buildings, or mountains. For this reason, one often assumes that the energy consumption is $||uv||^\delta$, for a suitable $\delta \in [2, 5]$.

- Energy consumption for communication along a path: Suppose one sends a message along a path $p = v_0, v_1, \dots, v_\ell$. Then the energy consumption is

$$\|p\|^\delta := \sum_{i=1}^{\ell} \|v_{i-1}v_i\|^\delta$$

For example, suppose one sends a message along a path p of ℓ nodes v_0, v_1, \dots, v_ℓ that are placed with unit distance on a line. Then the energy consumption is

$$\|p\|^\delta = \sum_{i=1}^{\ell} \|v_{i-1}v_i\|^\delta = \ell ,$$

whereas sending the message directly from v_0 to v_ℓ requires energy $\|v_0v_\ell\|^\delta = \ell^\delta$. Thus, sending a message via intermediate nodes can significantly reduce the energy consumption!

1.2 Geometric spanners and power spanners

We will study an overlay architecture for wireless networks that is based on so-called geometric spanners.

Definition 1.1. Let $V \subset \mathbb{R}^2$ be finite and $c \geq 1$ be a constant. Let $G = (V, E)$ be a graph.

1. G is called a geometric c -spanner if for all nodes $u, v \in V$ there exists a path p from u to v with $\|p\| \leq c \cdot \|uv\|$.
2. G is a (c, δ) -power spanner if for all nodes $u, v \in V$ there exists a path p from u to v in G with $\|p\|^\delta \leq c \cdot \|uv\|^\delta$.

If for all $\delta \geq 2$ there exists a constant c so that G is a (c, δ) -power spanner, then we simply call G a power spanner.

The following theorem shows that it suffices to show that a graph is a geometric c -spanner, for any constant c , in order to show that the graph is a power spanner.

Theorem 1.2. Each geometric c -spanner is a (c^δ, δ) -power spanner.

Proof. For every pair of nodes u and w , the geometric c -spanner contains a path $p = (u = v_0, v_1, \dots, v_\ell = w)$ such that $\|p\| \leq c \cdot \|uw\|$. This path satisfies the property needed for the (c^δ, δ) -power spanner, that is,

$$\|p\|^\delta = \sum_{i=1}^{\ell} \|v_{i-1}v_i\|^\delta \stackrel{(*)}{\leq} \left(\sum_{i=1}^{\ell} \|v_{i-1}v_i\| \right)^\delta \leq c^\delta \cdot \|uw\|^\delta ,$$

where the inequality marked with (*) follows from *Jensen's inequality*: For every convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, it holds $f(x) + f(y) \leq f(x + y)$, for every $x, y \in \mathbb{R}$. \square

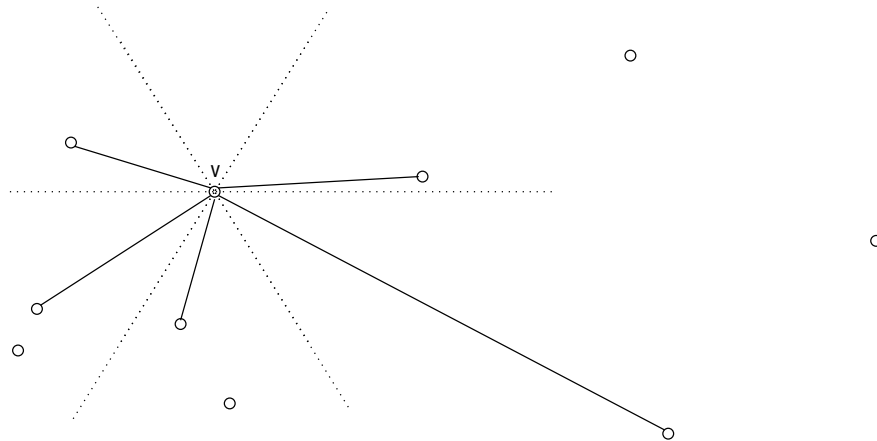


Figure 1: The outgoing edges of node v in a Yao graph $G_6(V)$.

1.3 Overlay networks based on the Yao graph

One way to build an overlay architecture for wireless communication network is to use the so-called Yao graph. We will show that this graph is a geometric c -spanner for some suitable c and, hence, also a power spanner.

Definition 1.3 (Yao graph). Let $V \subset \mathbb{R}^2$ be finite and $k \in \mathbb{N}$. Suppose the space around every node $v \in V$ is partitioned into k sectors of angle $\alpha = \frac{2\pi}{k}$. The Yao graph $G_k(V)$ consists of the following edges: Every node $u \in V$ has an outgoing edge into each non empty sector. The edge connects u to a closest node in this sector.

An example for the outgoing edges of a node is shown in Figure 1. In order to compute the Yao graph in a distributed fashion, each node only needs to find the closest other nodes in each of the k sectors.

The Yao graph also comes with a quite natural *routing algorithm*: For $u, w \in V$, let $C_{u,w}$ denotes the sector of node u in which w is located. If node u wants to send a message to a node w , it forwards the message along the outgoing edge leading to its neighbor v in $C_{u,w}$, and v continues in the same fashion until the message reaches w . The following lemma shows that this algorithm terminates as the Euclidian distance of the message from its destination shrinks whenever it is forwarded, provided the number of sectors is not too small, i.e., $k \geq 7$.

Lemma 1.4. Suppose we are given a point p in a Yao graph and two points q, r in the same sector of p with $\|pq\| \leq \|pr\|$. Then

$$\|qr\| \leq \|pr\| - \left(1 - 2 \cdot \sin\left(\frac{\alpha}{2}\right)\right) \cdot \|pq\| .$$

In particular, for $k \geq 7$, $\|qr\| < \|pr\|$, so that the routing algorithm is guaranteed to terminate.

Proof. For the analysis we add a further point q' that lies on the straight line between p and r with $\|pq\| = \|pq'\|$ as illustrated in Figure 2. We can make three observations:

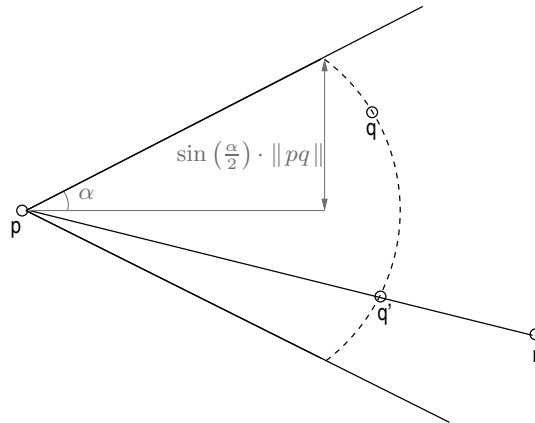


Figure 2: The situation described in the proof of Lemma 1.4.

1. $\|qr\| \leq \|qq'\| + \|q'r\|$,
2. $\|qq'\| \leq 2 \cdot \sin\left(\frac{\alpha}{2}\right) \cdot \|pq\|$, and
3. $\|q'r\| = \|pr\| - \|pq'\| = \|pr\| - \|pq\|$.

Combining these three equations yields

$$\begin{aligned} \|qr\| &\leq 2 \cdot \sin\left(\frac{\alpha}{2}\right) \cdot \|pq\| + \|pr\| - \|pq\| \\ &= \|pr\| - \left(1 - 2 \cdot \sin\left(\frac{\alpha}{2}\right)\right) \cdot \|pq\| \end{aligned}$$

□

The lemma does not only imply that the routing algorithm terminates, we also use it in order to show that the Yao graph is a geometric spanner.

Theorem 1.5. *If $k \geq 7$ then the Yao graph is a geometric c -spanner with $c = \frac{1}{1 - 2 \cdot \sin(\frac{\alpha}{2})}$.*

Proof. Consider any two nodes u and w . Let $p = (u = v_0, v_1, \dots, v_\ell = w)$ be the path from u to w as chosen by the routing algorithm. Lemma 1.4 gives,

$$\sum_{i=0}^{\ell-1} \|v_{i+1}w\| \leq \sum_{i=0}^{\ell-1} \left(\|v_iw\| - \left(1 - 2 \cdot \sin\left(\frac{\alpha}{2}\right)\right) \cdot \|v_iv_{i+1}\| \right)$$

Thus, for the length of the path, it follows

$$\begin{aligned}
 \|p\| &= \sum_{i=0}^{\ell-1} \|v_i v_{i+1}\| \\
 &\leq \frac{1}{1 - 2 \cdot \sin\left(\frac{\alpha}{2}\right)} \left(\sum_{i=0}^{\ell-1} \|v_i w\| - \|v_{i+1} w\| \right) \\
 &= \frac{1}{1 - 2 \cdot \sin\left(\frac{\alpha}{2}\right)} (\|v_0 w\| - \|v_\ell w\|) \\
 &= \frac{1}{1 - 2 \cdot \sin\left(\frac{\alpha}{2}\right)} \|uw\|
 \end{aligned}$$

□

Combining Theorem 1.2 with Theorem 1.5 shows that the Yao graph is also a power spanner.

Corollary 1.6. *The Yao graph is a $\left(\left(1 - 2 \cdot \sin\left(\frac{\alpha}{2}\right)\right)^{-\delta}, \delta\right)$ -power spanner.*