Player-Specific Congestion Games

Algorithmic Game Theory
Player-Specific Congestion Games

What if players have different valuations for congestion of a resource? Players have different delay functions for each resource:

A player-specific congestion game is a tuple $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_{i,r})_{i \in \mathcal{N}, r \in \mathcal{R}})$ with

- $\mathcal{N} = \{1, \ldots, n\}$, set of players
- $\mathcal{R} = \{1, \ldots, m\}$, set of resources
- $\Sigma_i \subseteq 2^\mathcal{R}$, strategy space of player $i$
- $d_{i,r} : \{1, \ldots, n\} \rightarrow \mathbb{Z}$, delay function of resource $r$ for player $i$
What if players have different valuations for congestion of a resource? Players have different delay functions for each resource:

A *player-specific congestion game* is a tuple 
\[ \Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_{i,r})_{i \in \mathcal{N}, r \in \mathcal{R}}) \] with

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- \( d_{i,r} : \{1, \ldots, n\} \rightarrow \mathbb{Z} \), delay function of resource \( r \) for player \( i \)

For any state \( S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n \),

- \( n_r = \) number of players with \( r \in S_i \)
- \( d_{i,r}(n_r) = \) delay of resource \( r \) for player \( i \)
- \( \delta_i(S) = \sum_{r \in S_i} d_{i,r}(n_r) = \) delay of player \( i \)
Theorem

Player-specific congestion games do not possess pure Nash equilibria in general.

Theorem

It is NP-hard to decide whether a player-specific congestion game has a pure Nash equilibrium.

Proofs: Exercise
Singelton games

What about games with a restricted structure such as singleton games. Remember: In singleton games every strategy contains only one resource. That is \(|s| = 1\) for each strategy \(s \in \Sigma_i\) and each player \(i \in \mathcal{N}\).

**Theorem**

*Player-specific singleton congestion games do not admit a potential function.*

Proof: Game with three players \(\{1, 2, 3\}\) and three resources \(\{a, b, c\}\). Strategy set for each player \(\{\{a\}, \{b\}, \{c\}\}\). delay functions (defined by the delay values for congestion 1, 2, and 3, respectively):

- \(d_{1,a} : 10/10/10\), \(d_{1,b} : 2/3/4\), \(d_{1,c} : 1/4/5\)
- \(d_{2,a} : 1/4/5\), \(d_{2,b} : 10/10/10\), \(d_{2,c} : 2/3/4\)
- \(d_{3,a} : 2/3/4\), \(d_{3,b} : 1/4/5\), \(d_{3,c} : 10/10/10\)

There is a cycle of improvement steps:

\((b, a, a) \rightarrow (c, a, a) \rightarrow (c, c, a) \rightarrow (c, c, b) \rightarrow (b, c, b) \rightarrow (b, a, b) \rightarrow (b, aa)\).

Thus, there cannot be a potential function.
Singelton games

Theorem

*In player-specific singletone congestion games a pure Nash equilibrium always exists and can be computed in polynomial time.*

Proofs: By induction over the number of players.

$n = 1$: Every player specific game with one player has a pure Nash equilibrium (PNE).

$n + 1$ players: Idea: Remove one player, the corresponding $n$ player game has a PNE by induction hypothesis. Add the remove player to this state and let players play their best response. Show that every player moves at most once.
Proof continued

Let $\Gamma$ be the $n+1$ player game and $\Gamma'$ be the $n$ player game after removing one player. Let $S'$ be a PNE of $\text{Gamma'}$. Add the removed player and let him play his best response to obtain $S_0$.
Define for a state $S$: A resource is called low if $n_r(S) = n_r(S')$ and high otherwise. Note there is exactly one high resource $r^*$ in $S_0$ and $n_{r^*}(S_0) = n_{r^*}(S') + 1$. Observe the following facts:

- Only players on a high resource may have a improvement step. If players change, their old resource becomes a low resource, their new resource a high resource.
- In every state there is exactly one high resource.
- When playing his best response, a player chooses the best high resource, i.e., resource $r' = \text{argmin}_{i,r}(n_r(S_0) + 1)$. He never leaves this resource again.

Thus, every player moves at most once until a PNE is reached.
Player-Specific Congestion Games

Players have different weights.

A *player-specific congestion game* is a tuple

$$\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}}, (w_i)_{i \in \mathcal{N}})$$

with

- $\mathcal{N} = \{1, \ldots, n\}$, set of players
- $\mathcal{R} = \{1, \ldots, m\}$, set of resources
- $\Sigma_i \subseteq 2^\mathcal{R}$, strategy space of player $i$
- $d_r : \{1, \ldots, n\} \rightarrow \mathbb{Z}$, delay function of resource $r$
Player-Specific Congestion Games

Players have different weights.

A "player-specific congestion game" is a tuple
\( \Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}}, (w_i)_{i \in \mathcal{N}}) \) with

- \( \mathcal{N} = \{1, \ldots, n\} \), set of players
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- \( \Sigma_i \subseteq 2^\mathcal{R} \), strategy space of player \( i \)
- \( d_r : \{1, \ldots, n\} \rightarrow \mathbb{Z} \), delay function of resource \( r \)

For any state \( S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n \),

- \( w_r = \) sum of weights of players with \( r \in S_i \)
- \( d_r(w_r(S)) = \) delay of resource \( r \) for player \( i \)
- \( \delta_i(S) = \sum_{r \in S_i} d_r(w_r) = \) delay of player \( i \)
Theorem
*Weighted congestion games do not posses pure Nash equilibria in general.*

Theorem
*It is NP-hard to decide whether a weighted congestion games has a pure Nash equilibrium.*

Proofs: Exercise
Linear Delay functions

Theorem

Weighted congestion games with linear delay functions are potential games.

Proof:
Potential function:

$$\Phi(S) = \sum_{r \in R} w_r(S)d_r(w_s(S)) + \sum_{i \in N} w_i \cdot \sum_{r \in S_i} d_r(w_i)$$

Next: Show that $\Phi(S)$ decreases, each time a player makes an improvement step: Let $S = (S_1, \ldots, S_n)$ and $S'_j$ be a better response for a player $j$ and $S' = (S_{-j}, S'_j)$.
Show that $\Phi(S) - \Phi(S') > 0...$
\[
\Phi(S) - \Phi(S') = \sum_{r \in R} w_r(S) d_r(w_s(S)) + \sum_{i \in \mathcal{N}} w_i \cdot \sum_{r \in S_i} d_r(w_i) - \\
\sum_{r \in R} w_r(S') d_r(w_s(S')) - \sum_{i \in \mathcal{N}} w_i \cdot \sum_{r \in S_i'} d_r(w_i) = \\
\sum_{r \in S_j \setminus S_j'} (w_r(S) d_r(w_r(S)) - (w_r - w_j) d_r(w_r - w_j) + w_j d_r(w_j)) + \\
\sum_{r \in S_j' \setminus S_j} (w_r(S) d_r(w_r(S)) - (w_r + w_j) d_r(w_r + w_j) - w_j d_r(w_j)) = \\
\sum_{r \in S_j \setminus S_j'} (w_r(S)(a_r w_r(S) + b_r) - (w_r - w_j)(a_r(w_r - w_j) + b_r) + w_j(a_r(w_j) + b_r)) + \\
\sum_{r \in S_j' \setminus S_j} (w_r(S)(a_r w_r(S) + b_r) - (w_r + w_j)(a_r(w_r + w_j) + b_r) - w_j(a_r w_j + b_r)) = \\
\sum_{r \in S_j \setminus S_j'} (2w_j a_r w_r + 2w_j b_r) - \sum_{r \in S_j' \setminus S_j} (2w_j(a_r(w_r + w_j) + b_r)) = \\
2w_j \left( \sum_{r \in S_i} d_r(w_r) - \sum_{r \in S_i'} d_r(w_r + w_j) \right)
\]
The proof shows that the potential function $\Phi(S)$ decreases by $2w_j \cdot \delta$, each time a player makes an improvement step and decreases his delay by $\delta$. Thus, weighted congestion games with linear delay functions are potential games and always possess a pure Nash equilibrium.